Applications of Flows + Cuts

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Bipartite Matching

Vertex Disjoint Paths

Path Covers

Project Selection

Assignment Problems

Edge Orientation

Densest Subgraph

Playoff Elimination

scheduling

Bipartite Vertex Cover

(high-level comment)

"only two algorithms"*

1. identify subproblems
2. change the input

Applications of Flows + Cuts

Gameplan

1. Survey a bunch of problems
2. Start solving them one by one

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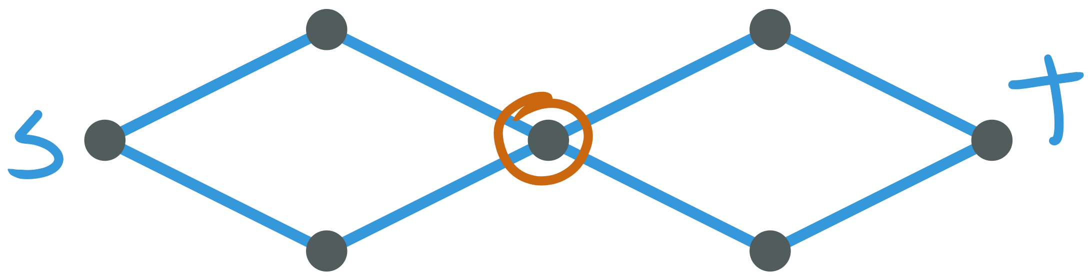
Vertex Disjoint Paths



2 edge disjoint paths

1 vertex disjoint path

vertex cut = 1

"vertex capacities" $c : V \to \mathbb{R}_{>0}$

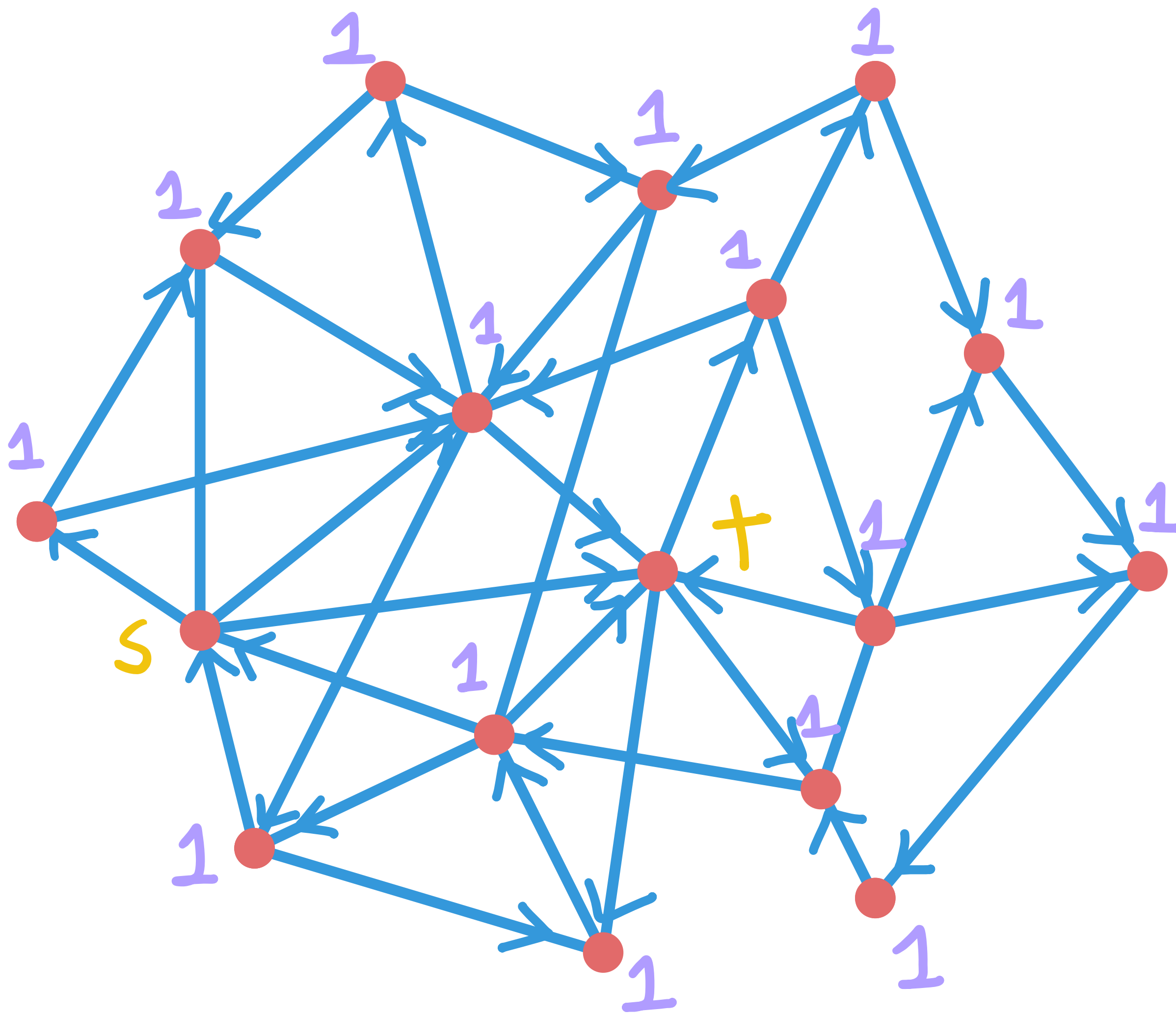
$c(v)$ = limit of flow through v

e.g. $c(v) = 1$

$\forall v \notin \{s,t\}$

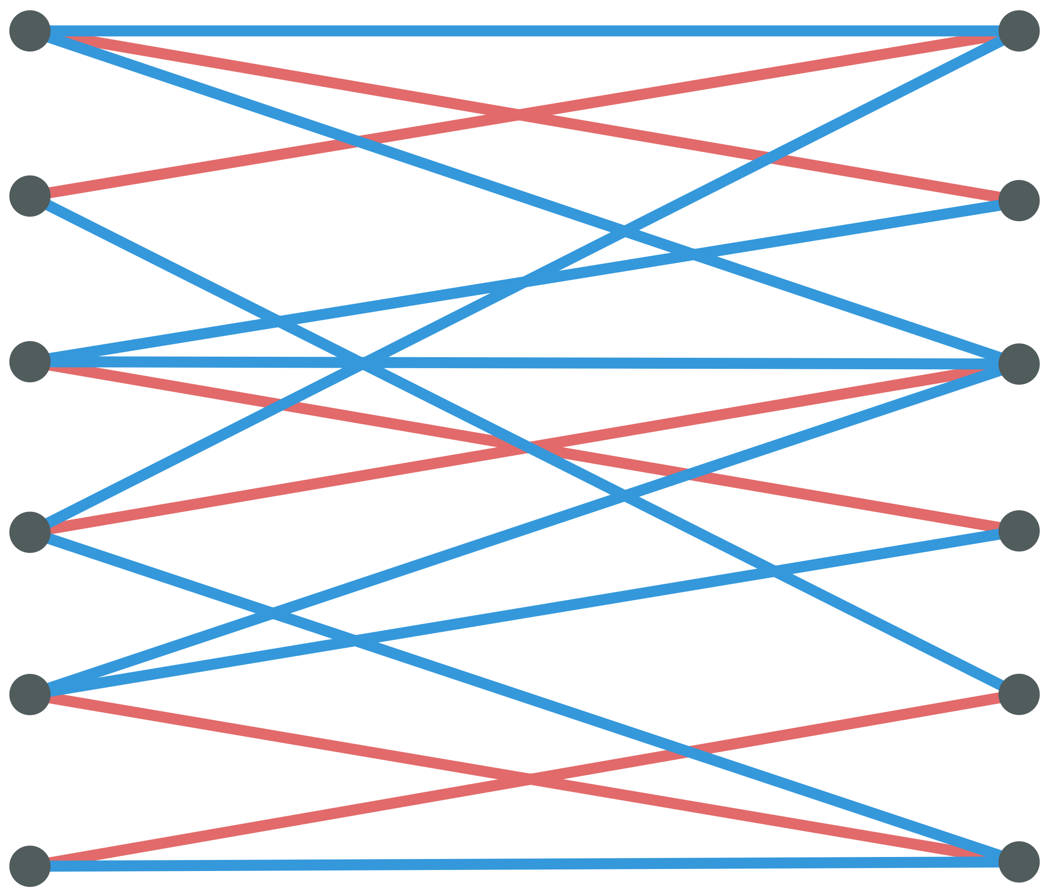
$\Rightarrow$ <u>vertex</u>

disjoint paths



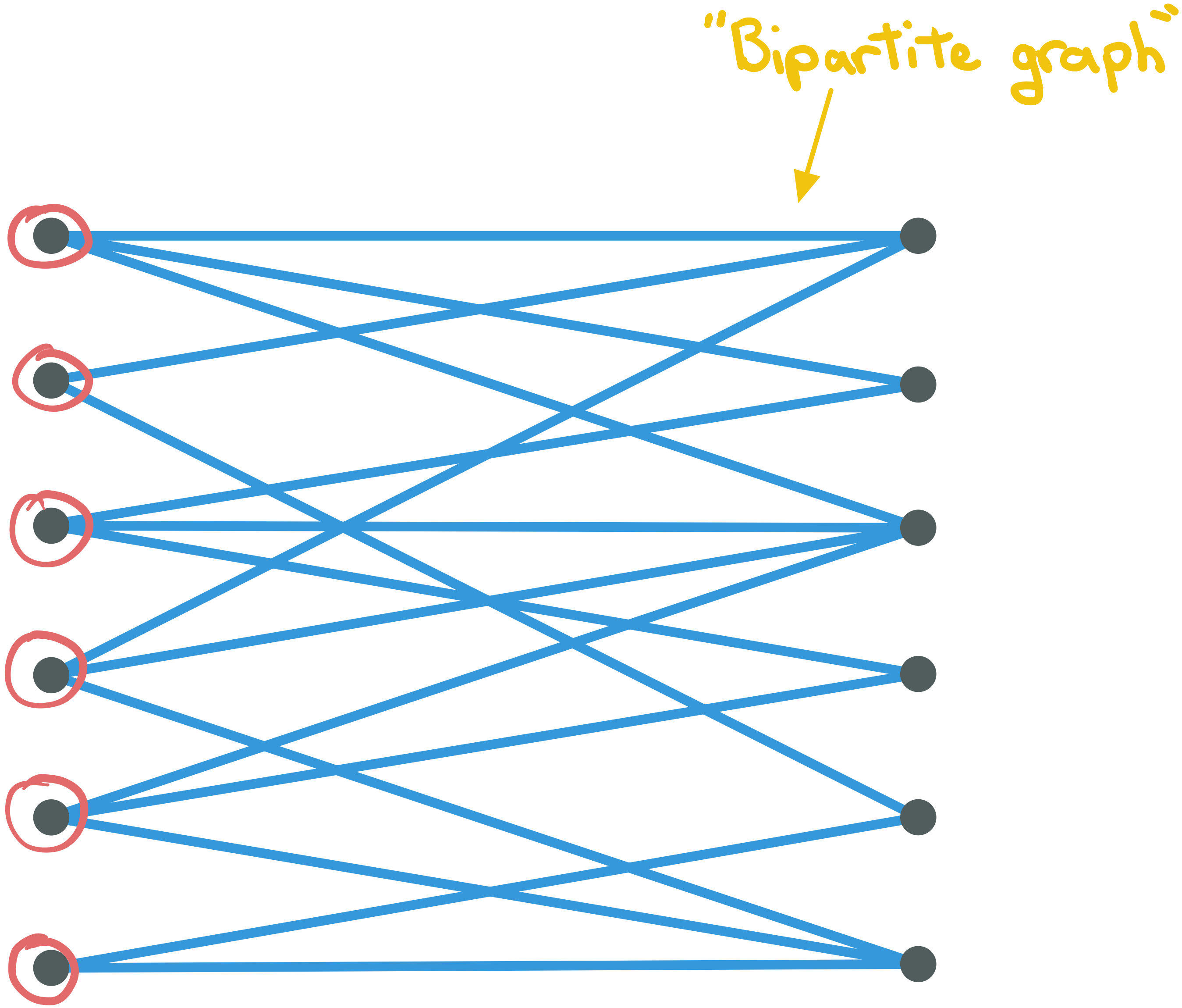
Bipartite Matching

"Bipartite graph"



Goal: max $\underbrace{\text{set of edges w/ distinct endpoints}}_{\text{"matching"}}$

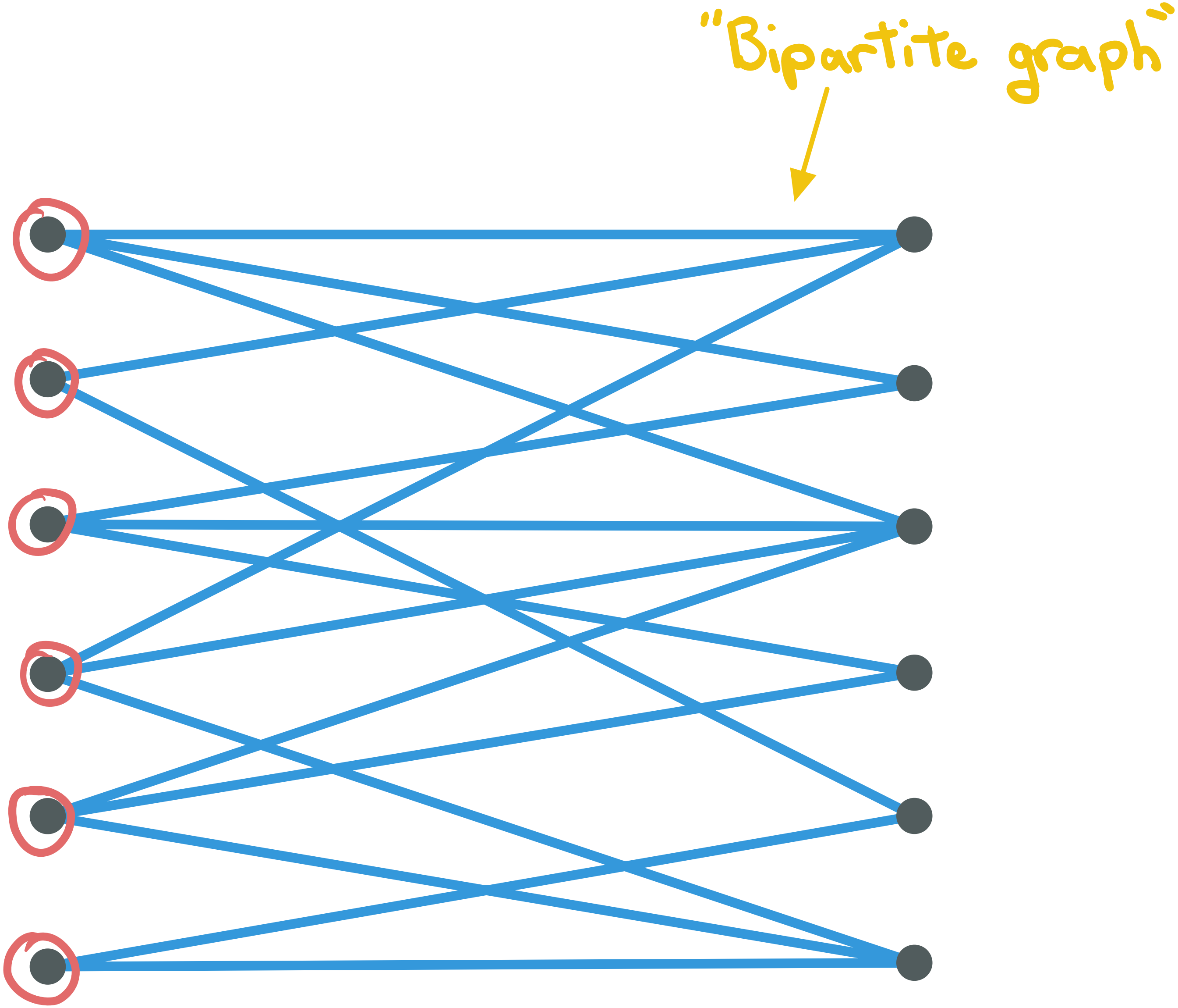
Bipartite Vertex Cover



Goal: min set of vertices incident to all edges

"vertex cover"

Bipartite Vertex Cover



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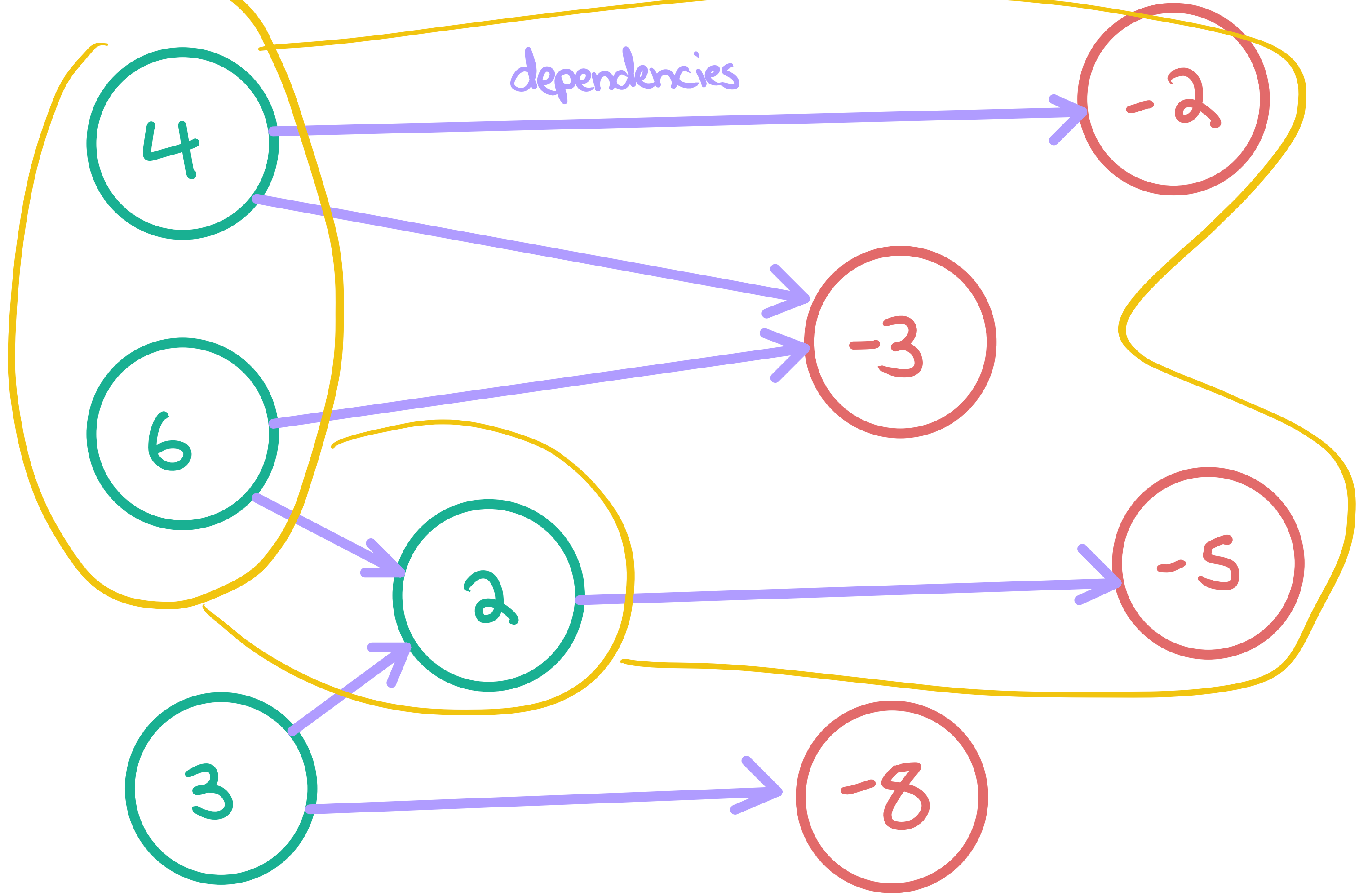
Project Selection



Projects have net profits (can be <0 for costs) and dependencies

Goal: select projects (including dependencies) w/ max total profit

Playoffs Elimination

Playoffs Elimination

can Boston catch the Yankees?

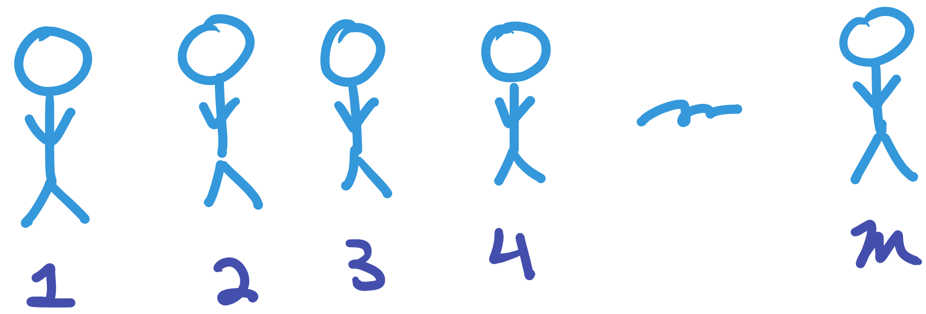
Team	Wins	Games left
New York	92	2
Baltimore	91	3
Toronto	91	3
Boston	90	2

Remaining schedule

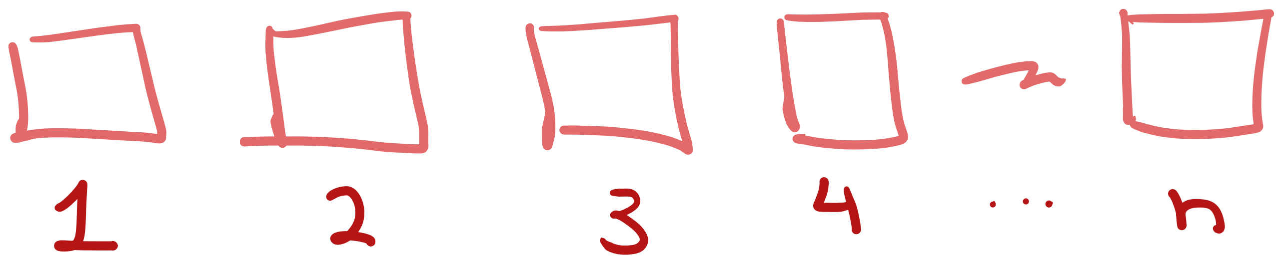
(wins)	(92)	(91)	(91)	(90)
	New York	Baltimore	Toronto	Boston
New York	×	~~1~~	~~1~~	0
Baltimore	1	×	1	~~1~~
Toronto	1	1	×	~~1~~
Boston	0	1	1	×

Assignment Problems

m individuals



n jobs



goal: fill as many jobs as possible w/ individuals

the catch: not everyone is qualified for all jobs

Assignment Problems

"Qualification matrix"

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

"1" = qualified

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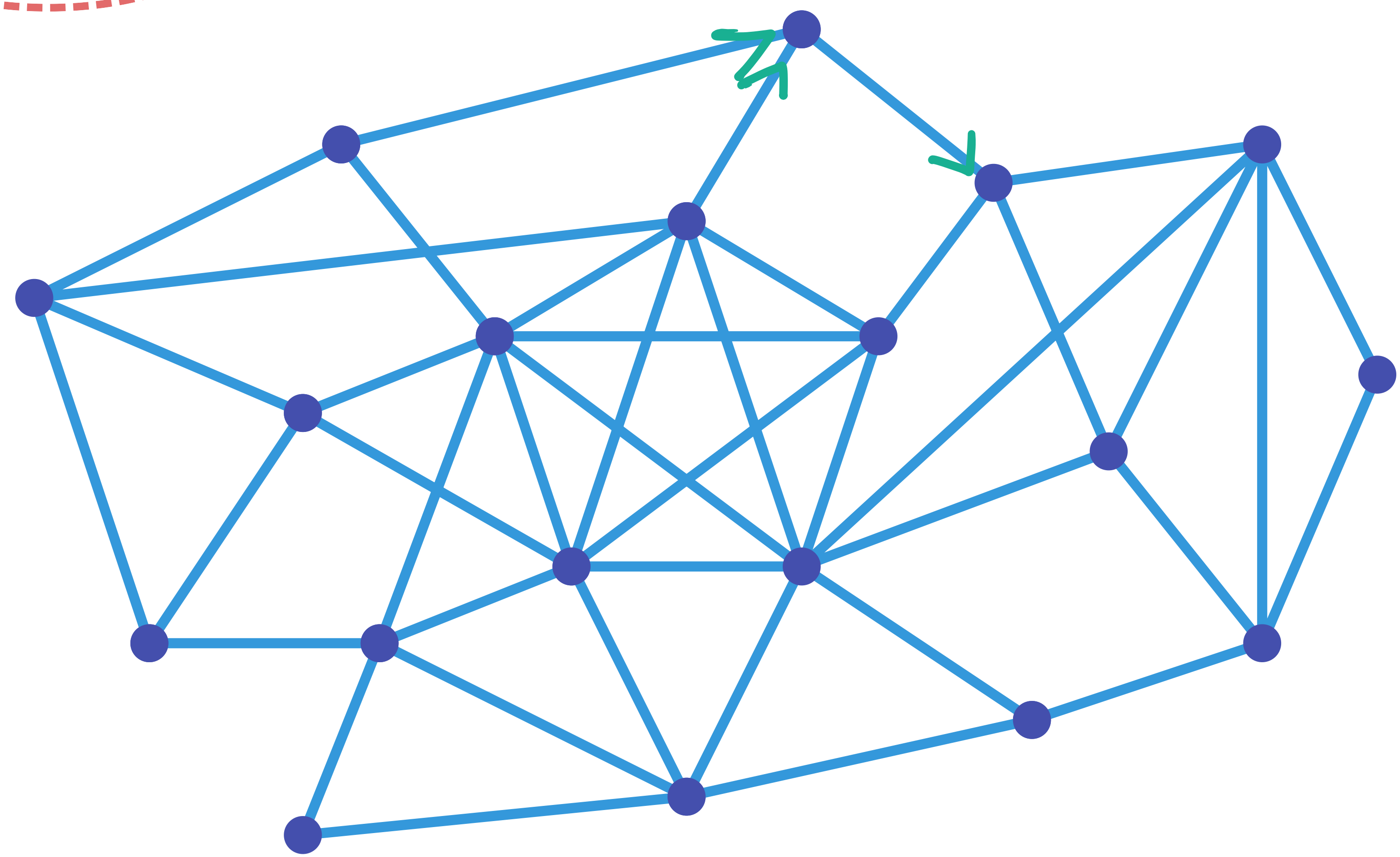
Edge Orientation

input: undirected graph



Goal: direct the edges to minimize the max in-degree

Edge Orientation

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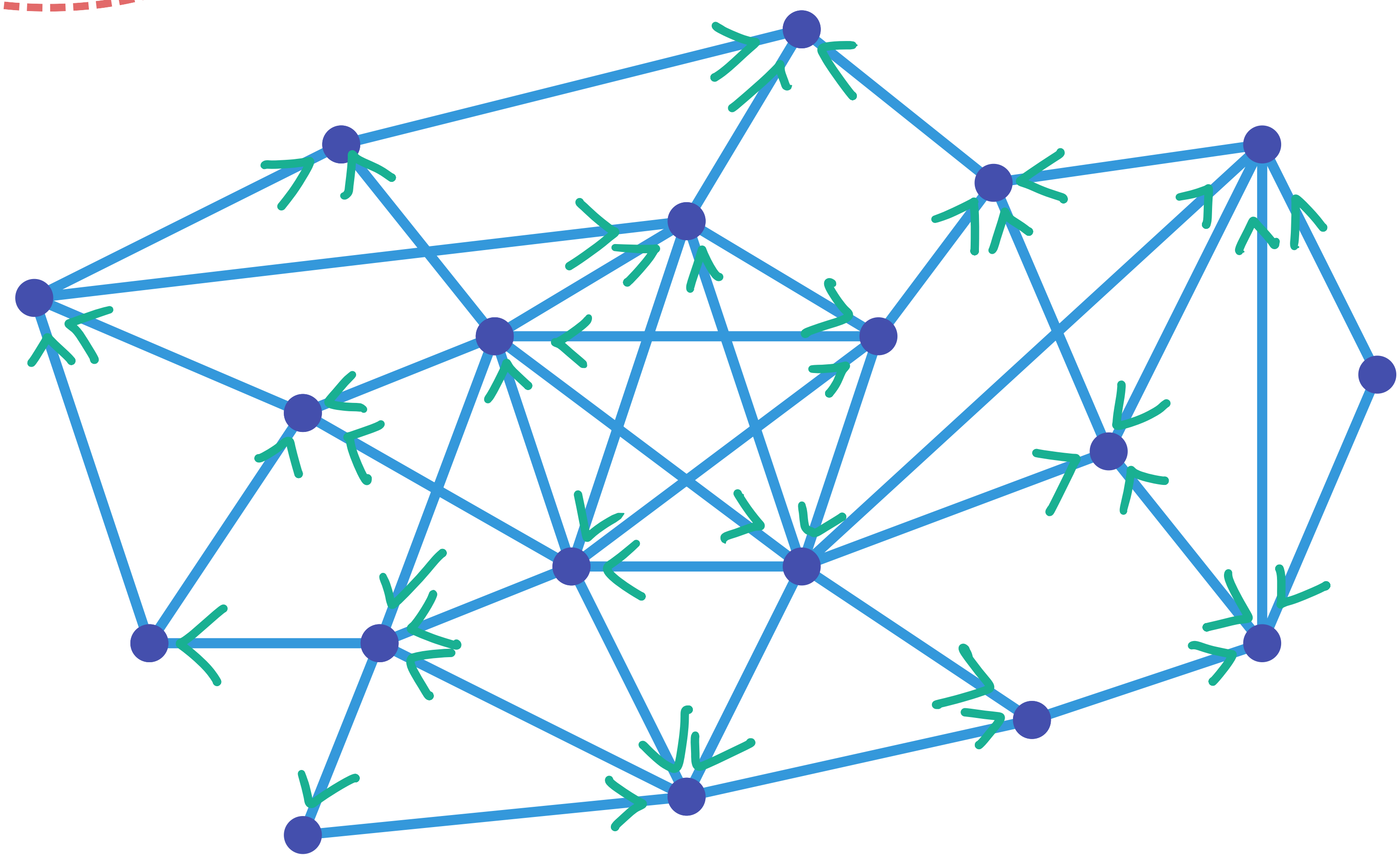

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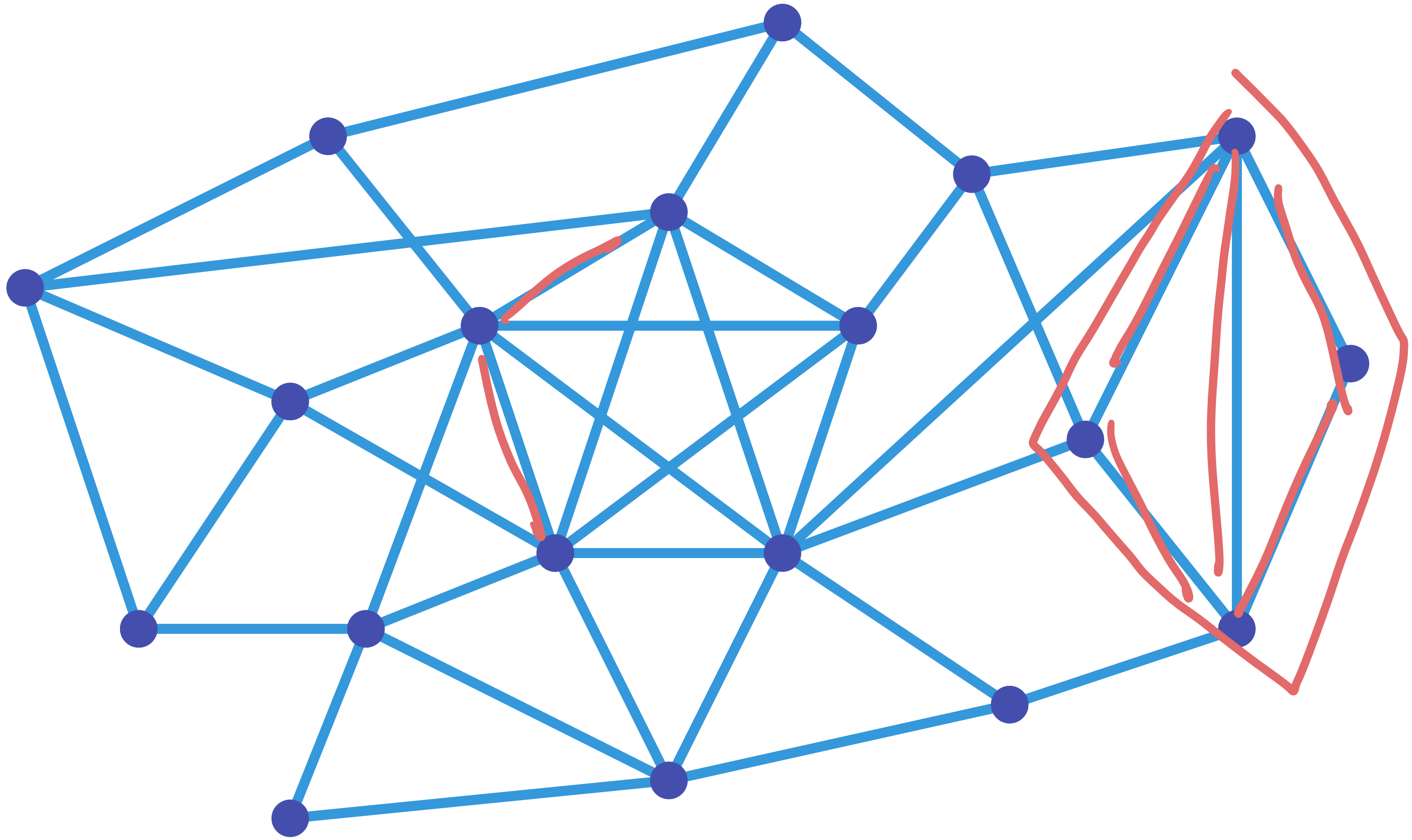
Densest Subgraph

"density" $= \dfrac{\#\text{ edges}}{\#\text{ vertices}}$



Goal: compute densest subgraph

Densest Subgraph

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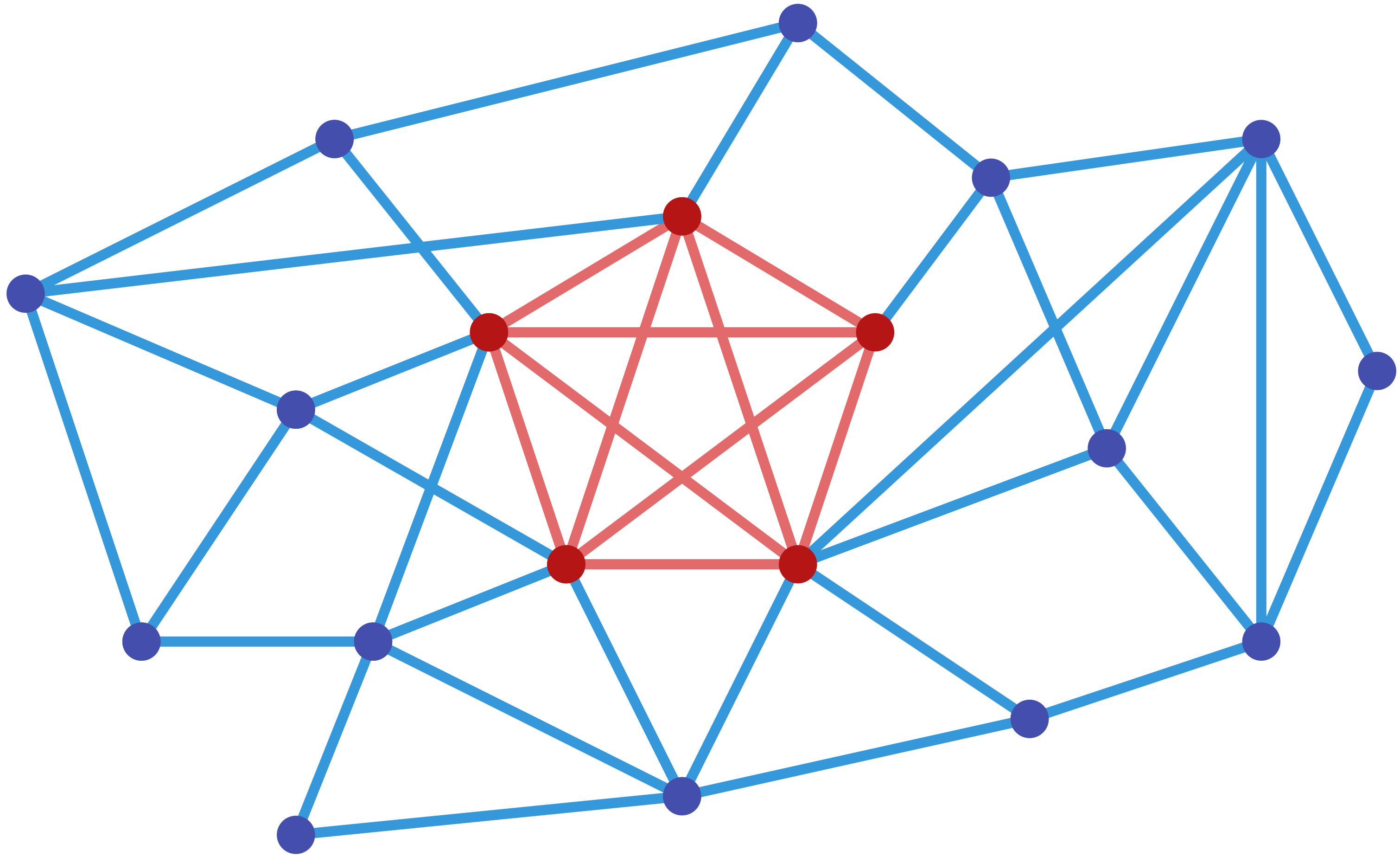

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Applications of Flows + Cuts

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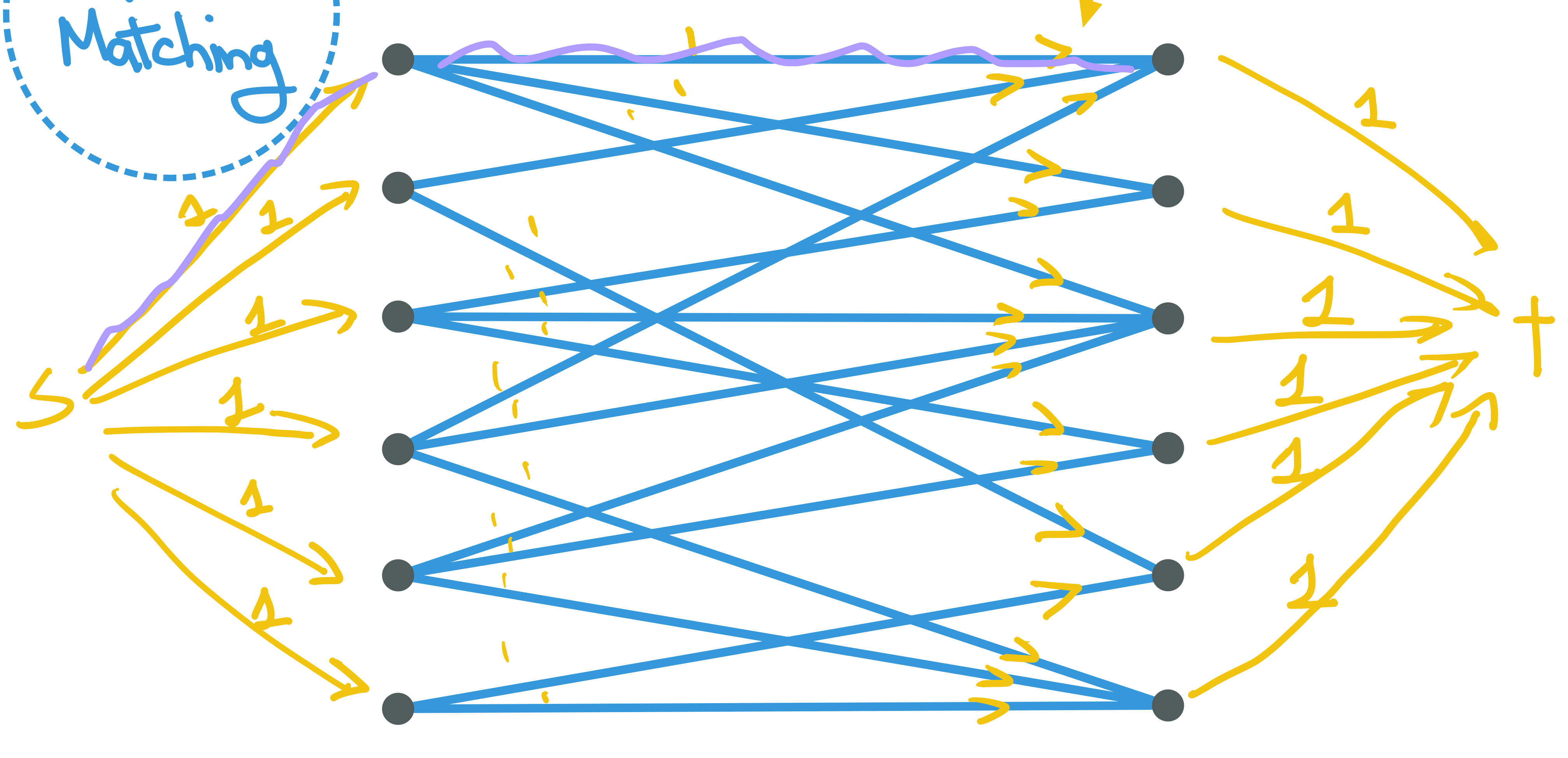
Bipartite Matching

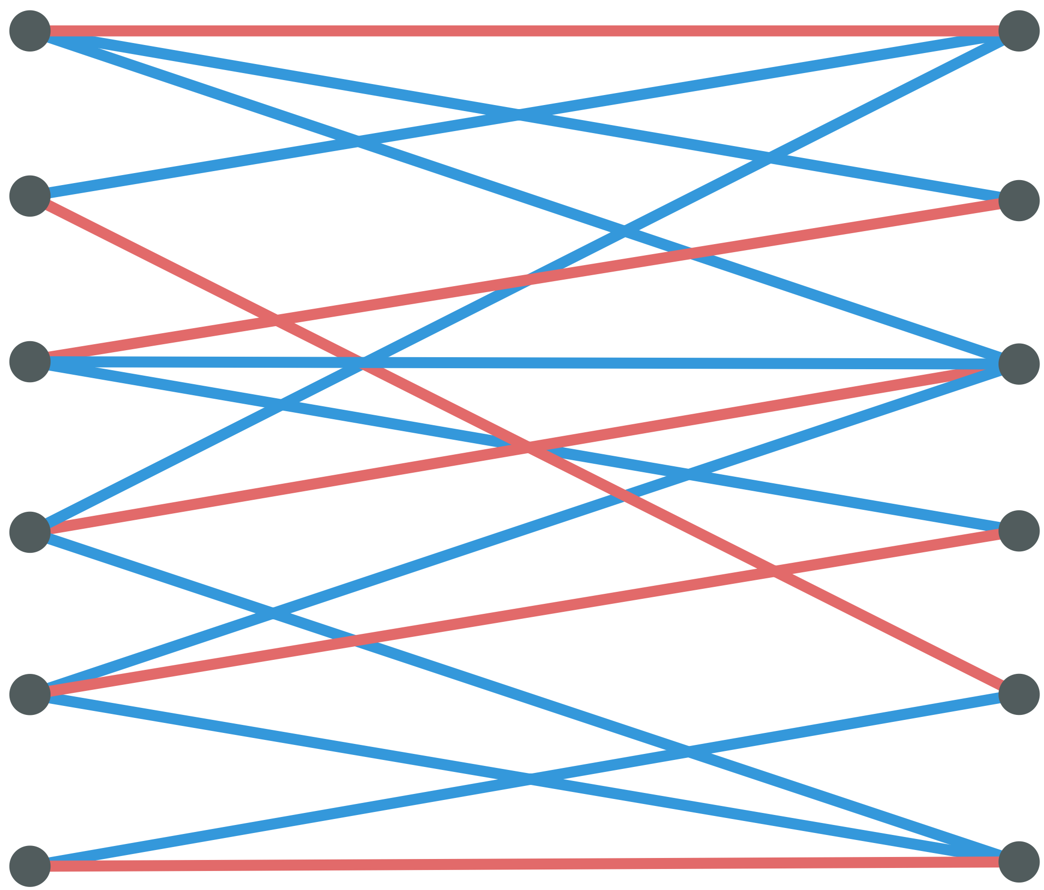


Goal: max set of edges w/ distinct endpoints

"matching"

Bipartite Matching

(bipartite graph)



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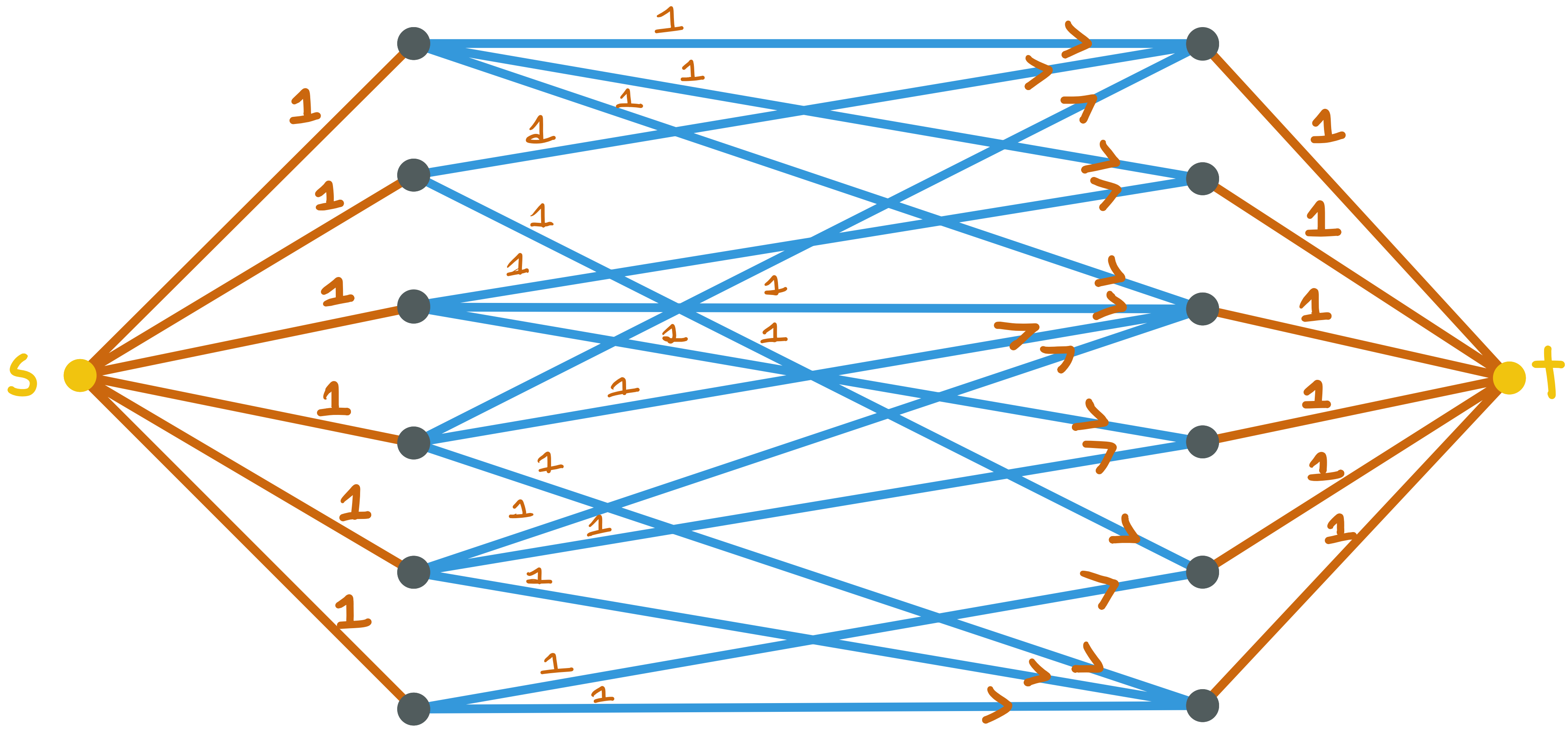
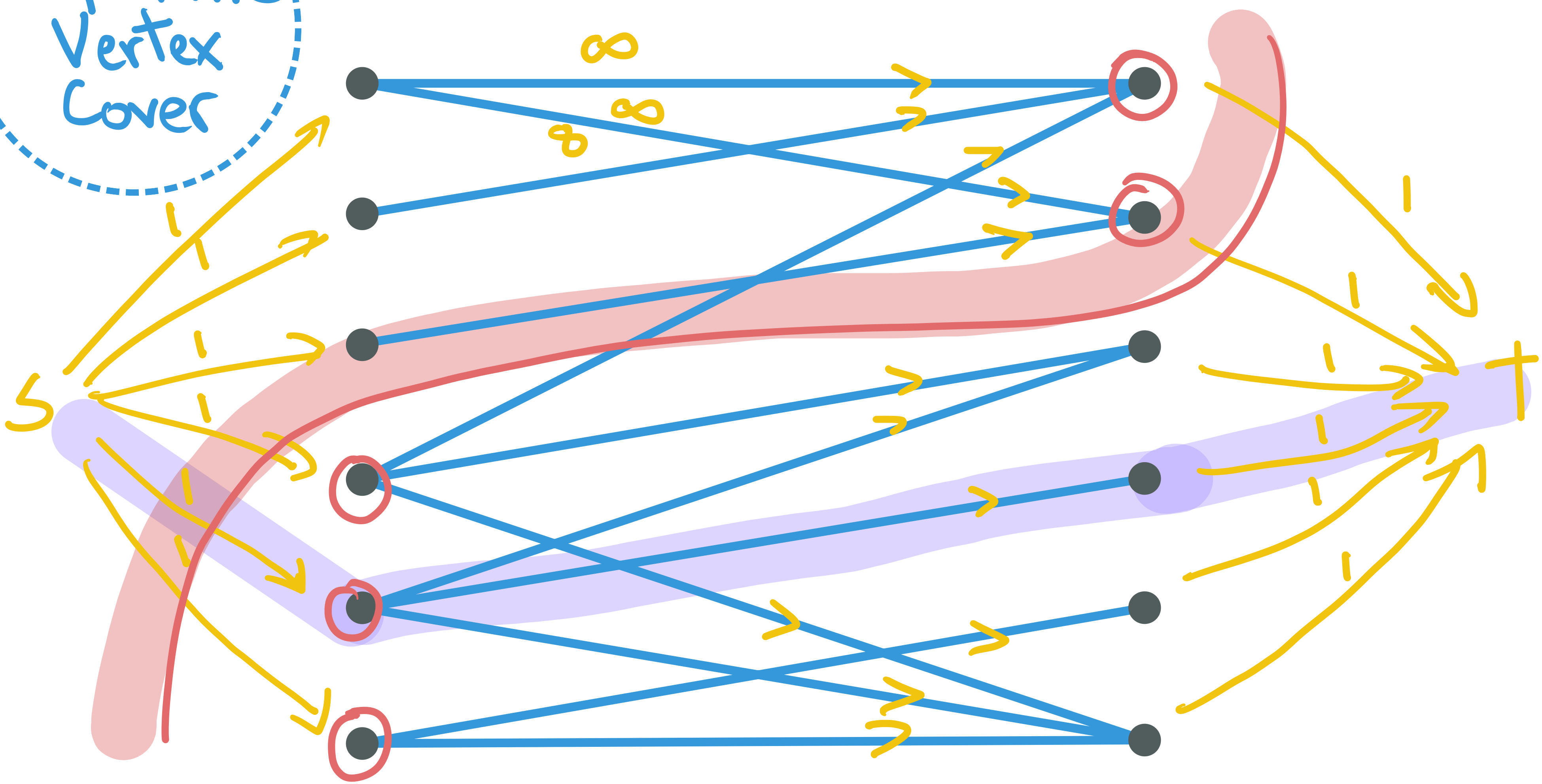


Matching w/ k edges $\longleftrightarrow$ k edge-disjoint (s,t)-paths

(see lecture notes for detailed proof)

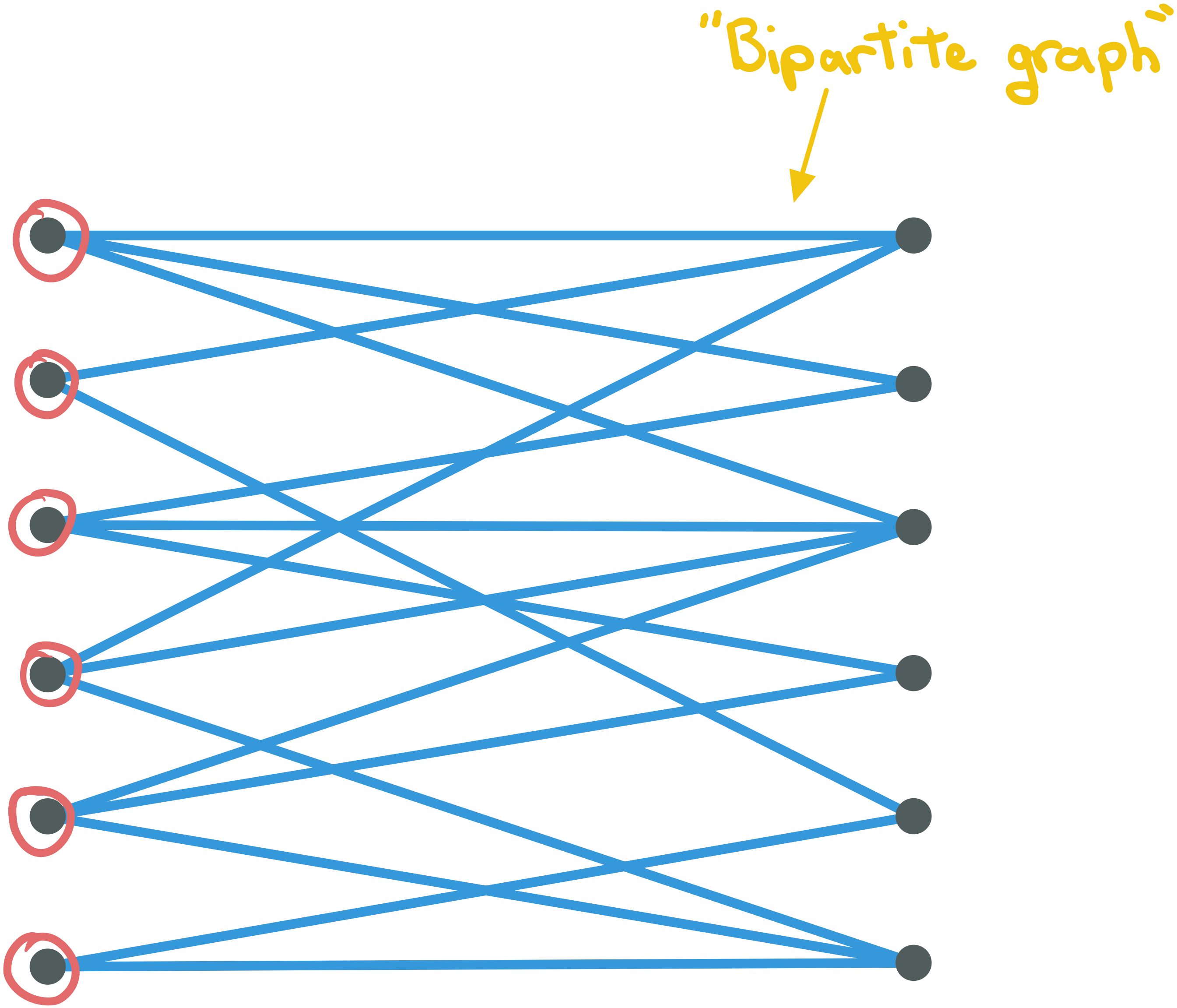
Bipartite Vertex Cover

(bipartite graph)



Goal: min $\underbrace{\text{set of vertices incident to all edges}}_{\text{"vertex cover"}}$

Bipartite Vertex Cover



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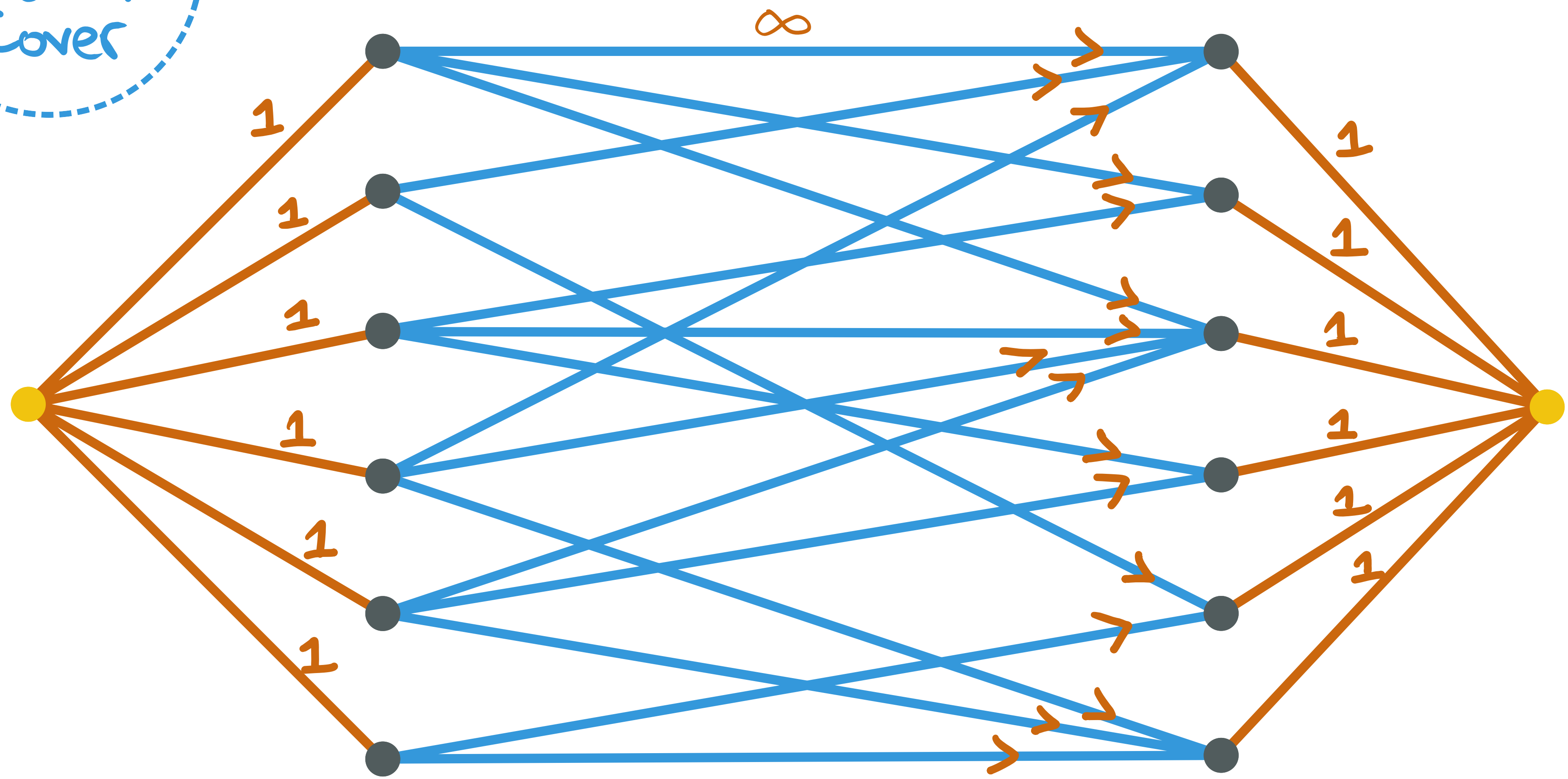
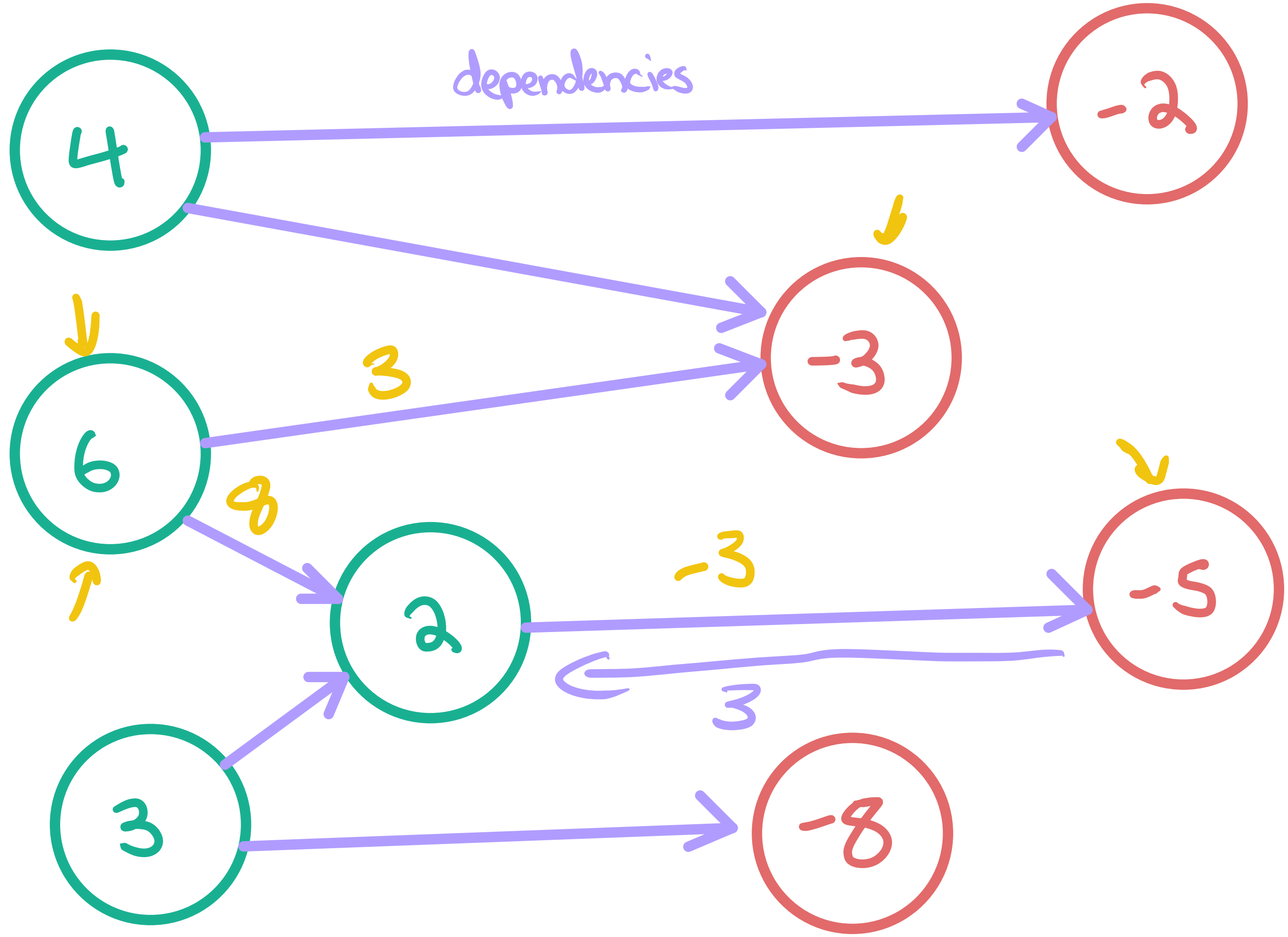
Bipartite Vertex Cover



(s,t)-cut of size k

vertex cover of size k

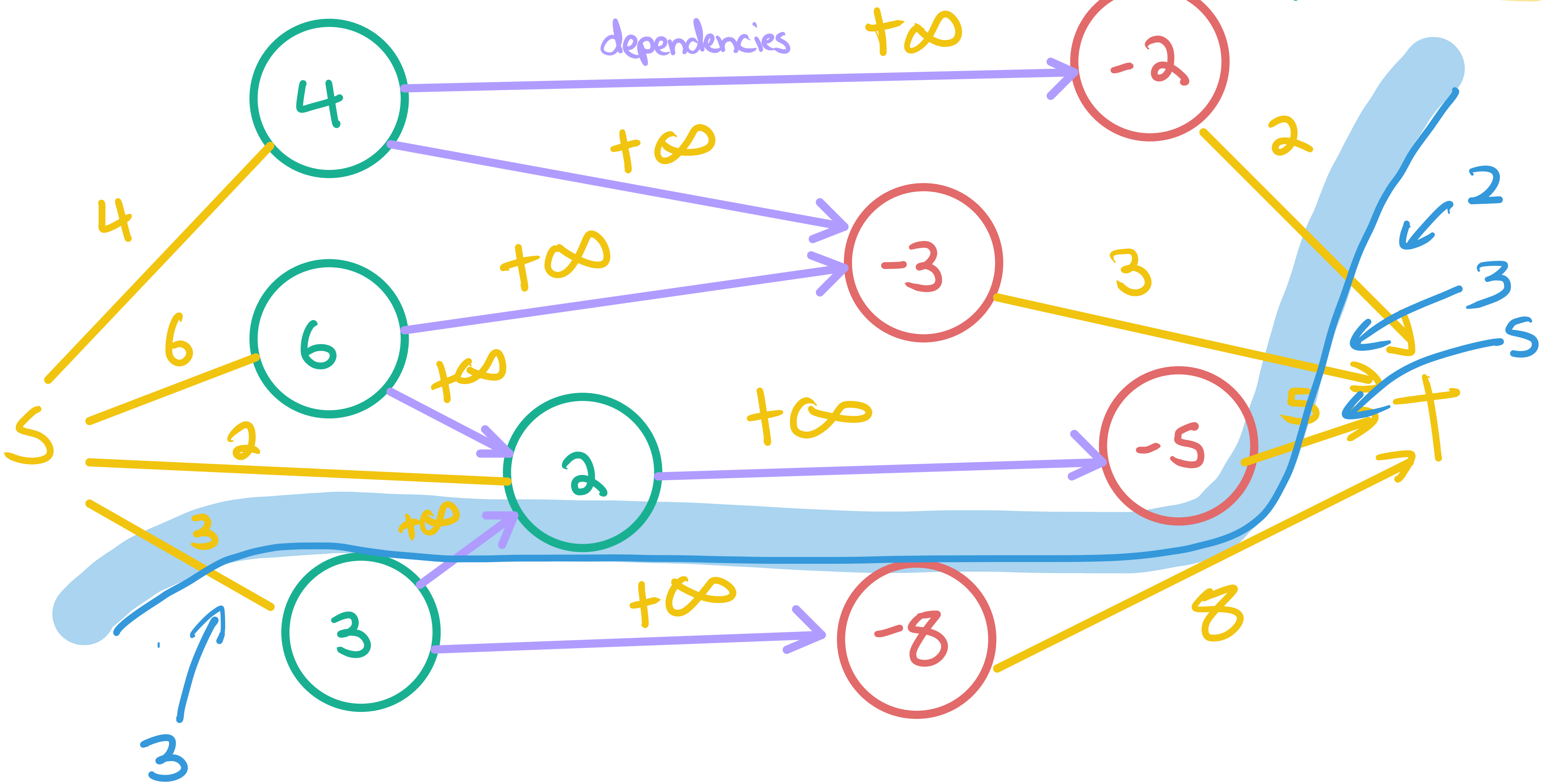
Project Selection



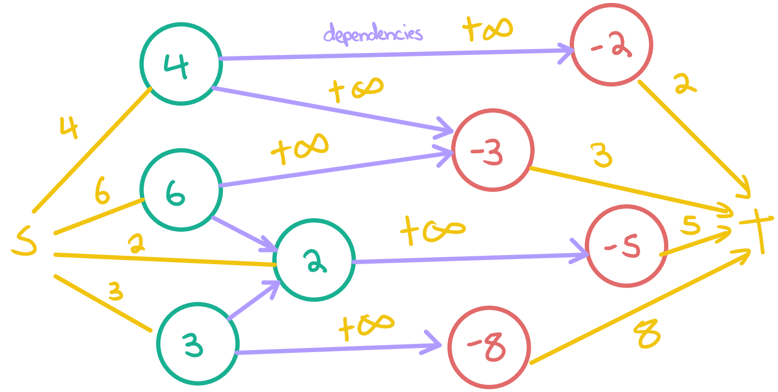
Projects have net profits (can be <0 for costs) and dependencies

Goal: select projects (including dependencies) w/ max total profit (cost cut) = (cost of bad projects taken) + (profit of good projects omitted) = (total profit of all)

(cost of bad projects taken) + − (profit of good projects taken)



claim: min-cut ⇒ best project selection



dependencies
s
t
4
6
2
3
-2
-3
-5
-8
+∞

Playoffs Elimination

Playoffs Elimination

can Boston catch the Yankees?

Team	Wins	Games left
New York	92	2
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Remaining schedule

(wins)	(92)	(91)	(91)	(90)
	New York	Baltimore	Toronto	Boston
New York	×	1	1	0
Baltimore	1	×	1	1
Toronto	1	1	×	1
Boston	0	1	1	×

(wins)	(92) New York	(91) Baltimore	(91) Toronto	(90) Boston
New York	×	1	1	0
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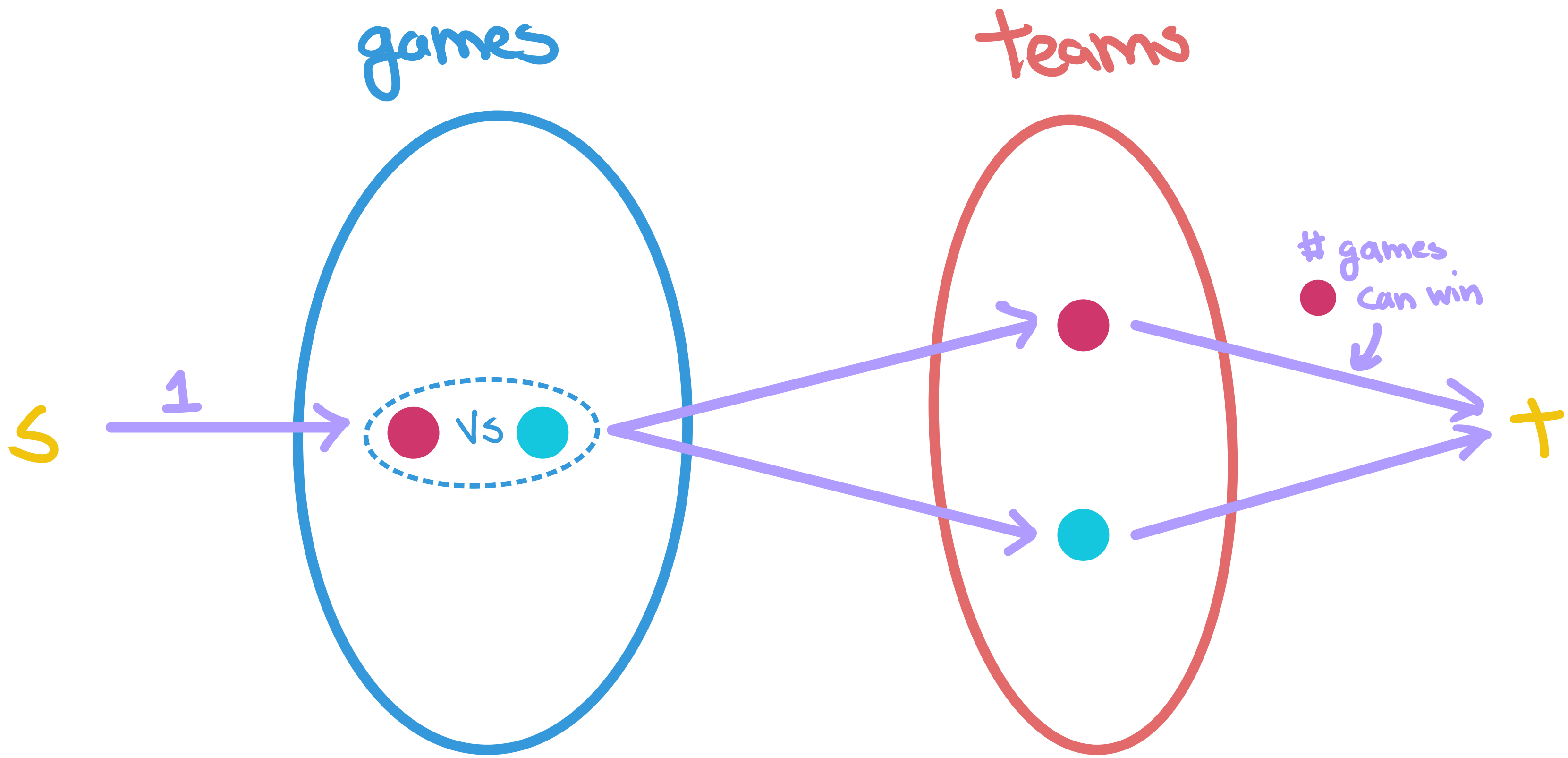
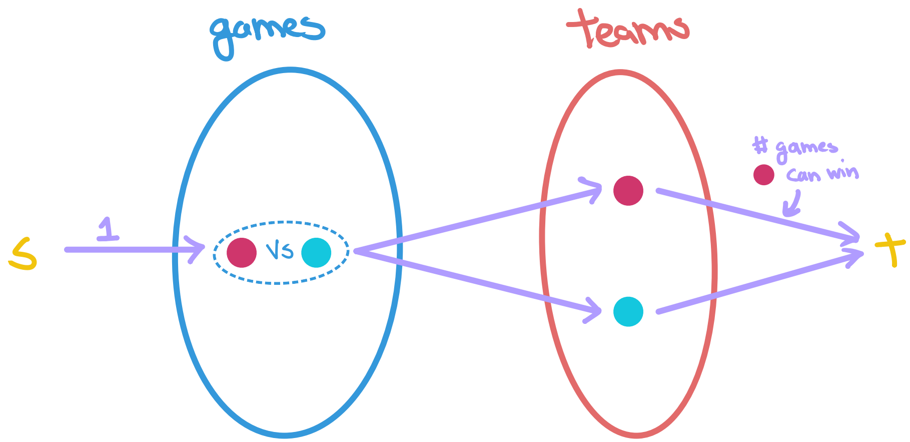


games
teams
s
1
vs
t
games
can win

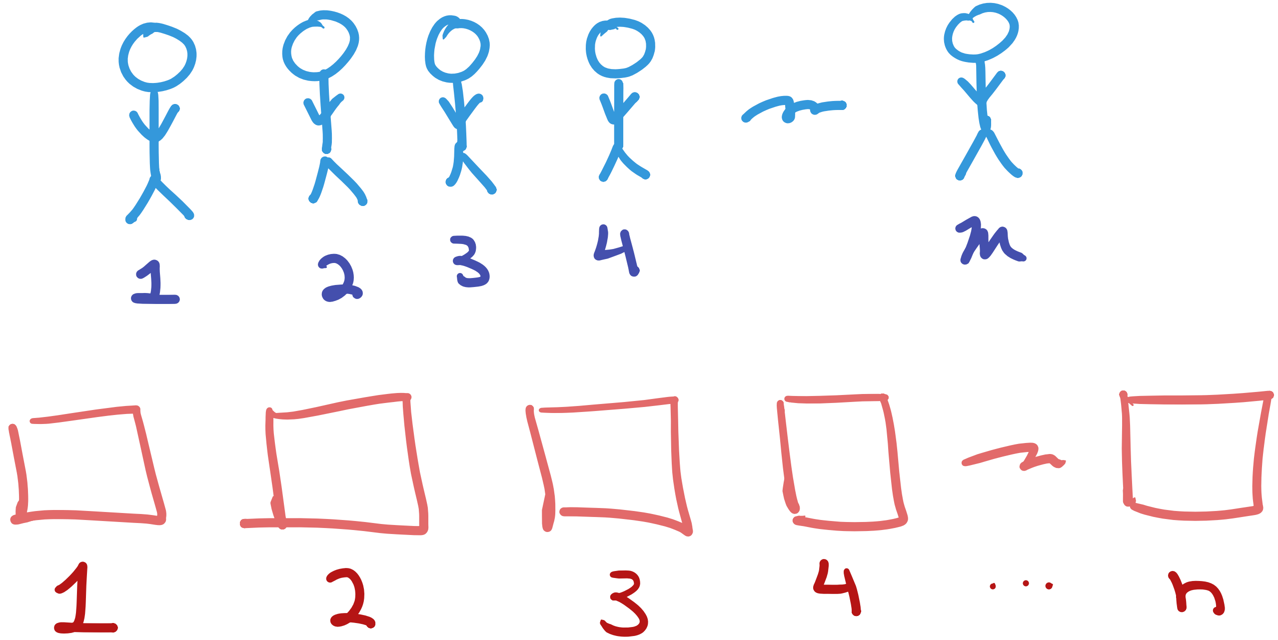

games
teams
s
1
vs
games can win
t


s
1
1
1
1
1
New York vs Baltimore
New York vs Toronto
Baltimore vs Toronto
Baltimore vs Boston
Toronto vs Boston
New York
Baltimore
Toronto
Boston
0
1
1
∞
t

Assignment Problems

m individuals

n jobs



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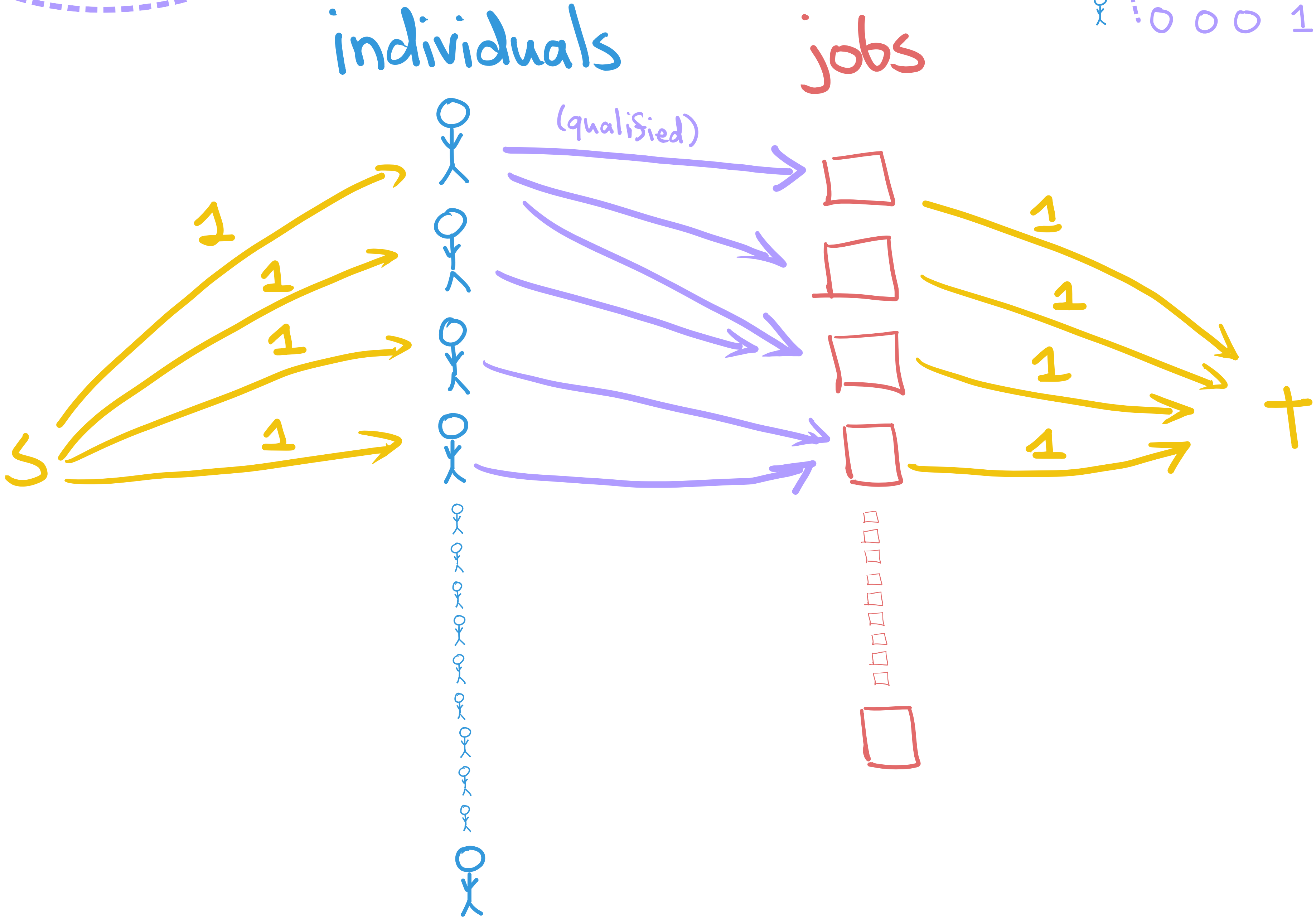
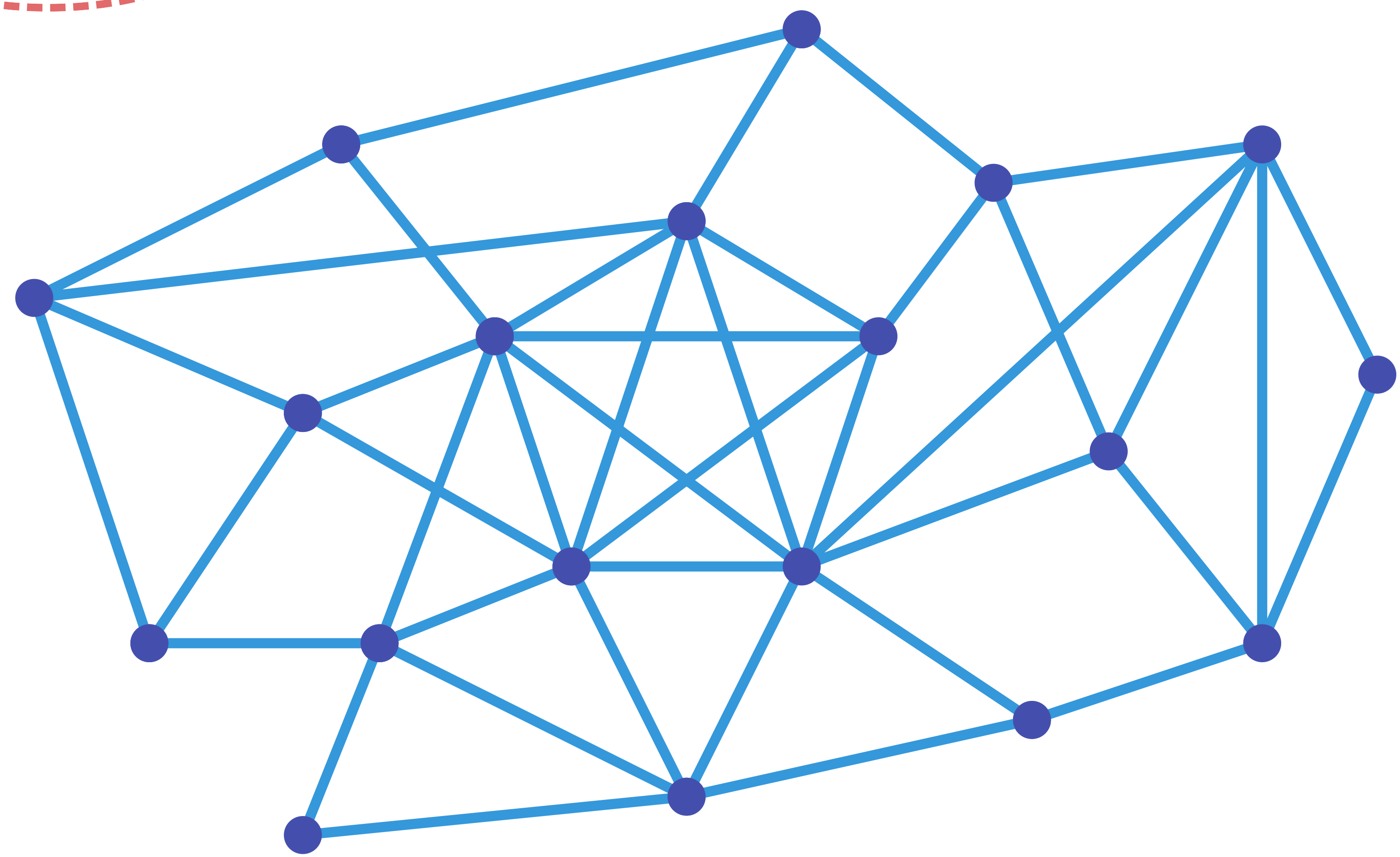


Edge Orientation

input: undirected graph



Goal: direct the edges to minimize the max density

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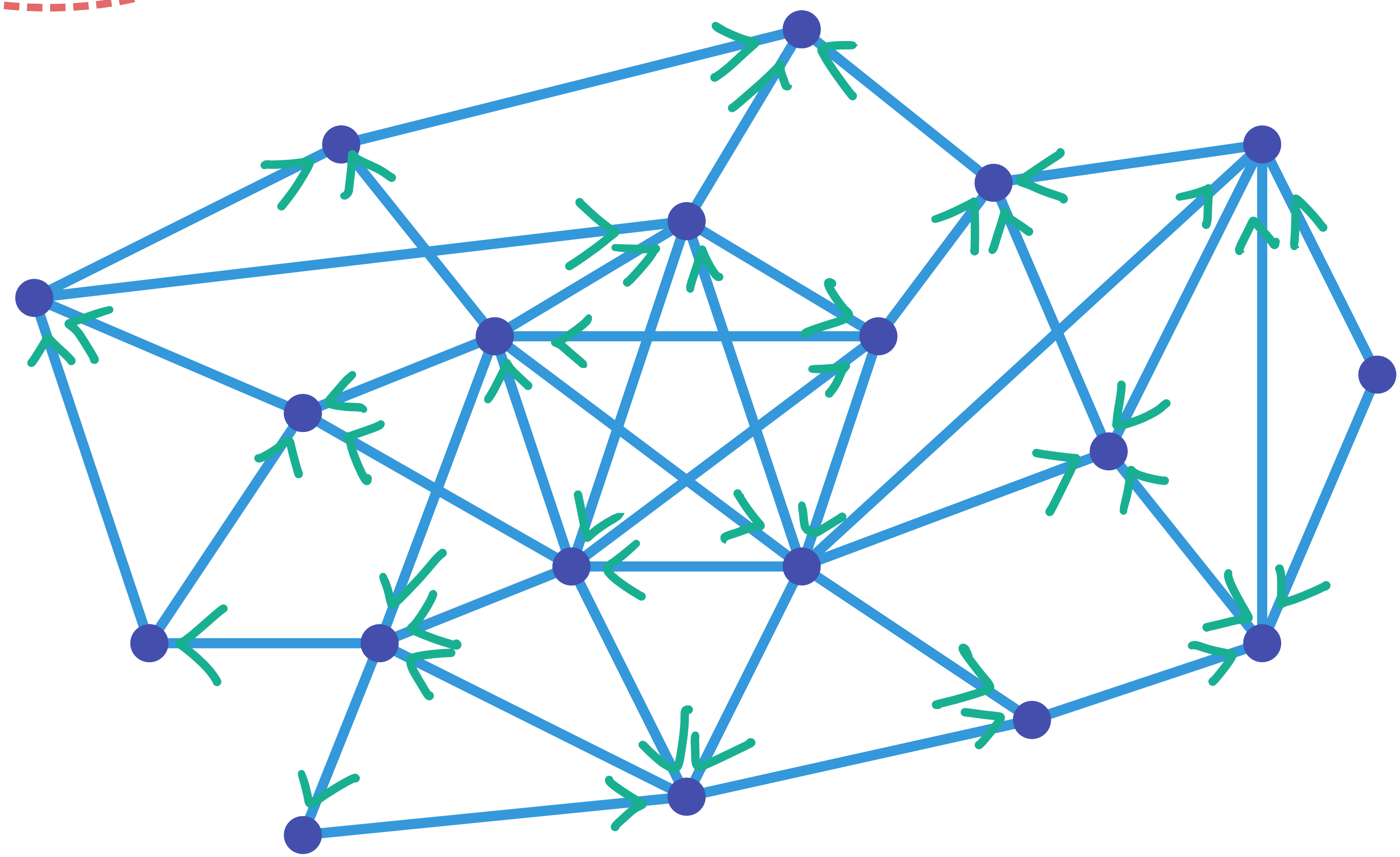

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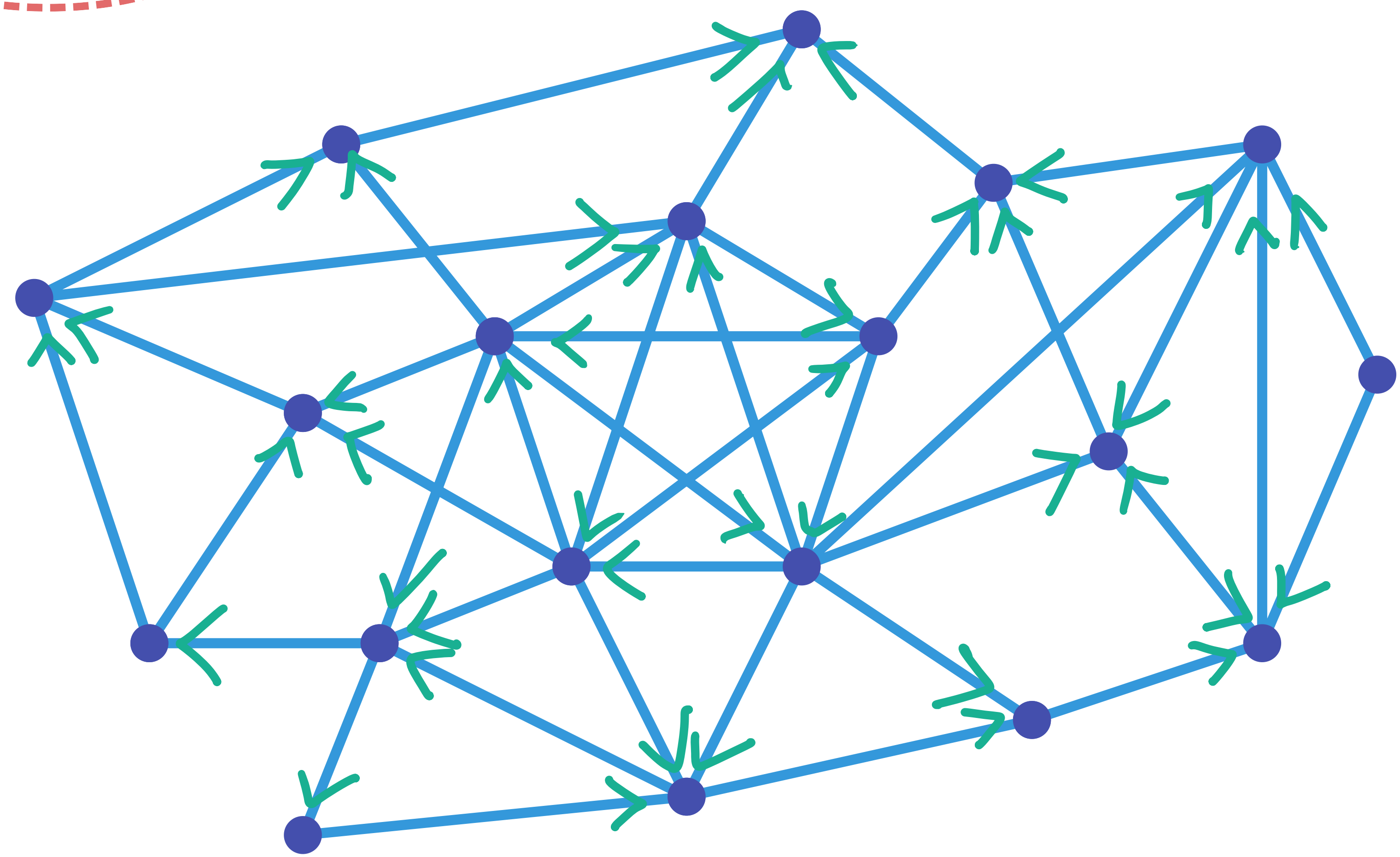
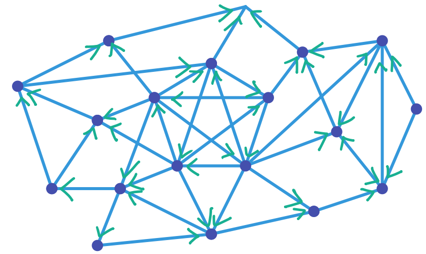
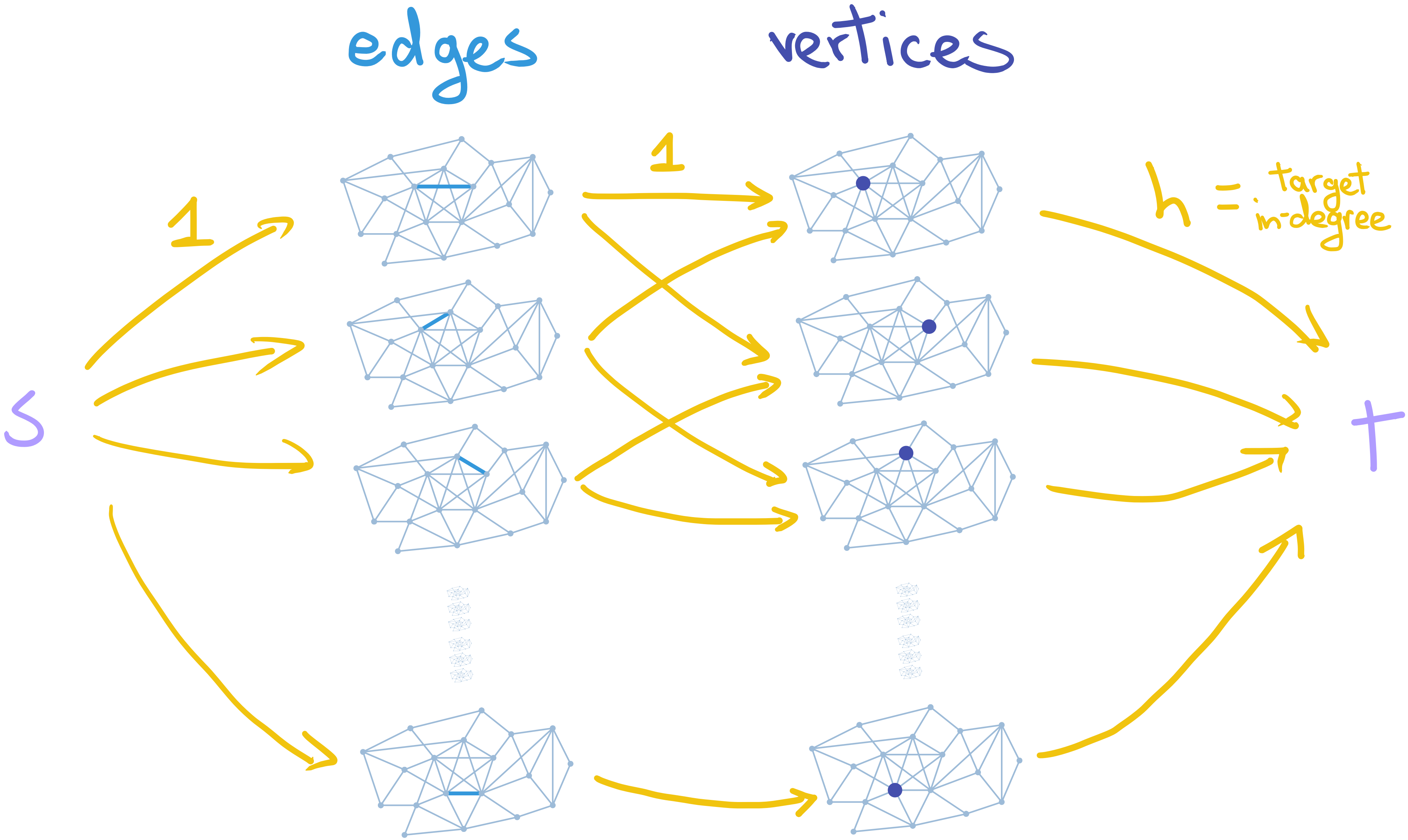


Decision version: given h>0, is there an orientation w/ max in-degree ≤h?

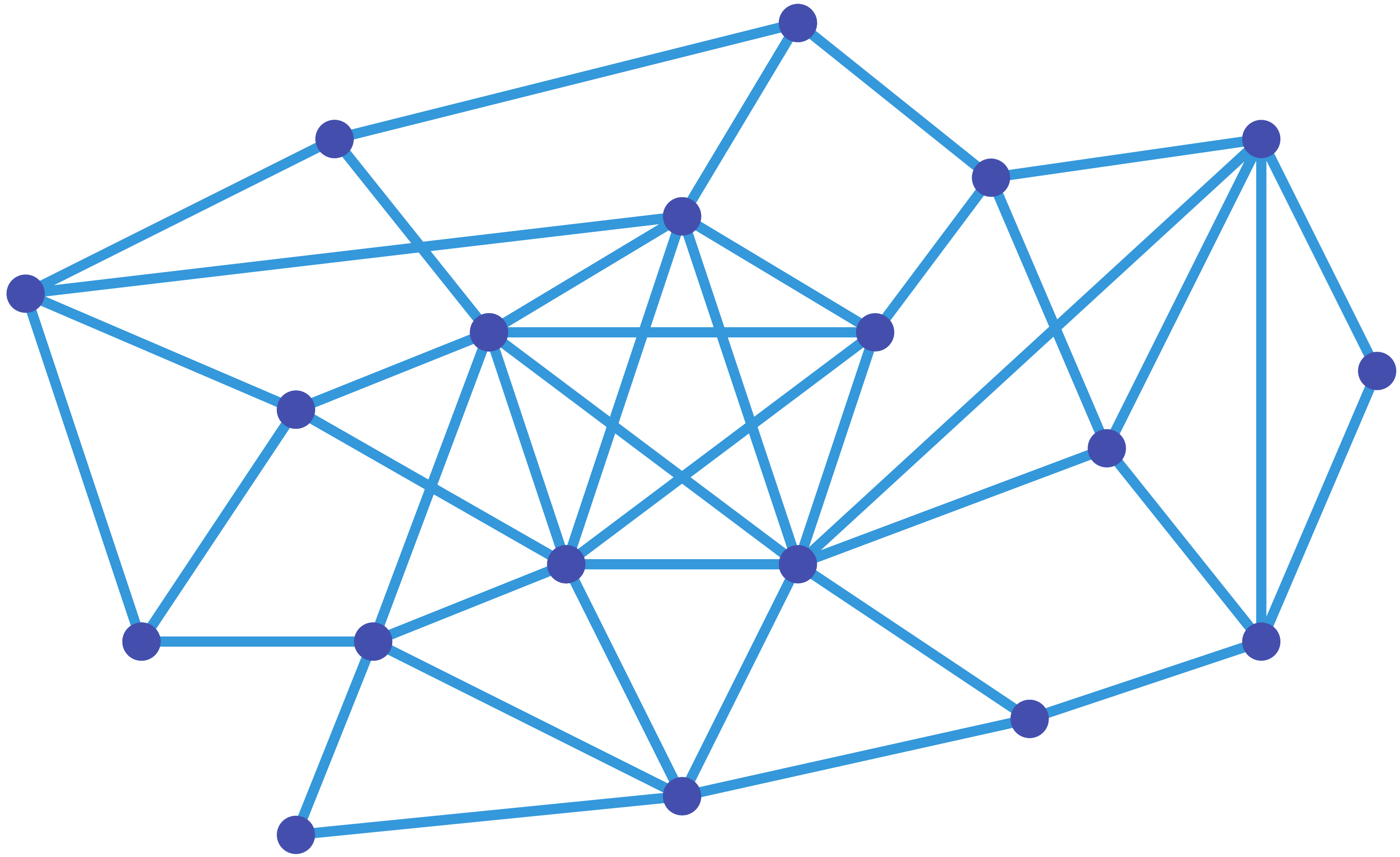
Edge Orientation





Densest Subgraph

"density" = $\frac{\text{\# edges}}{\text{\# vertices}}$



Goal: compute densest subgraph

Densest Subgraph

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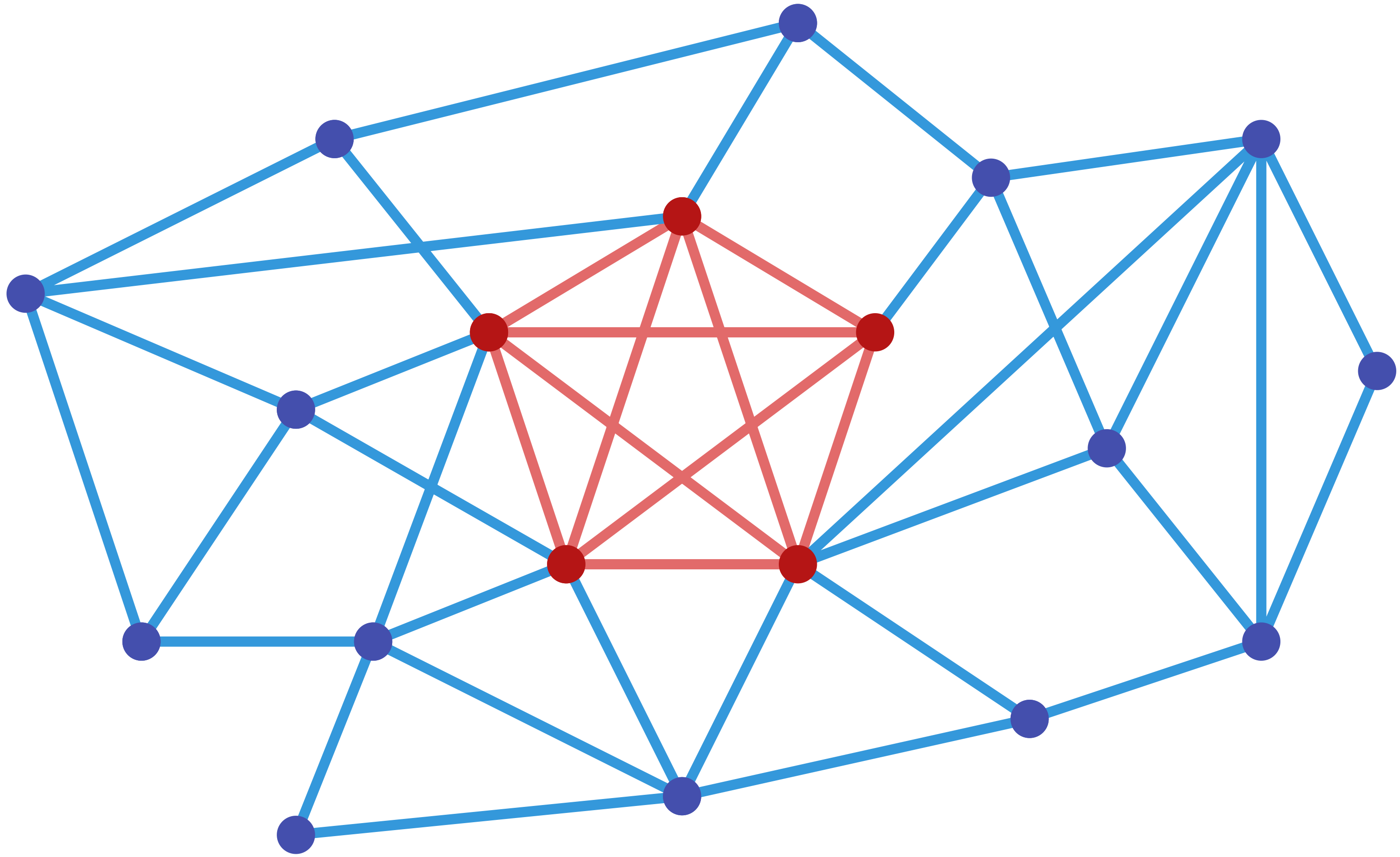

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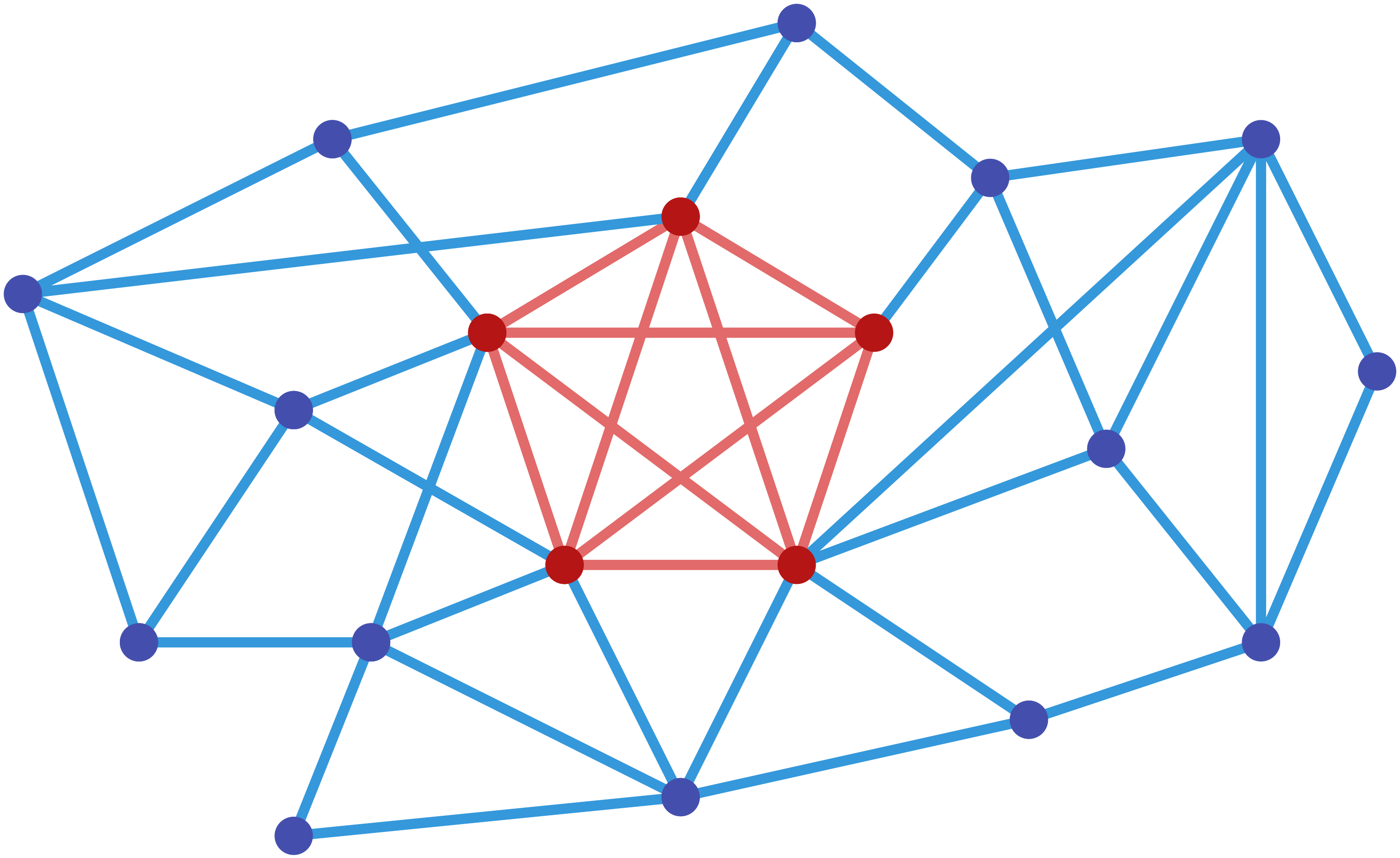
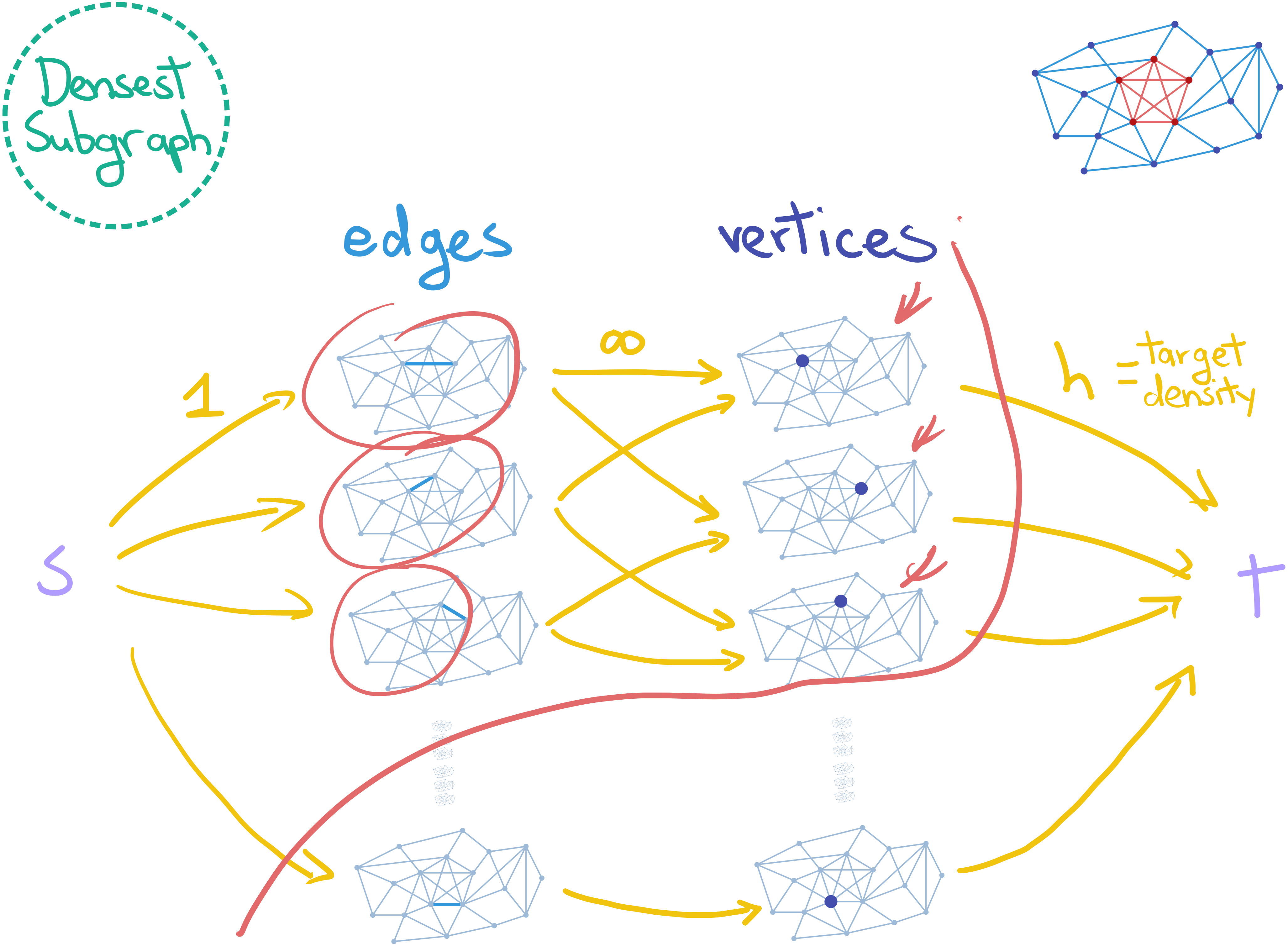
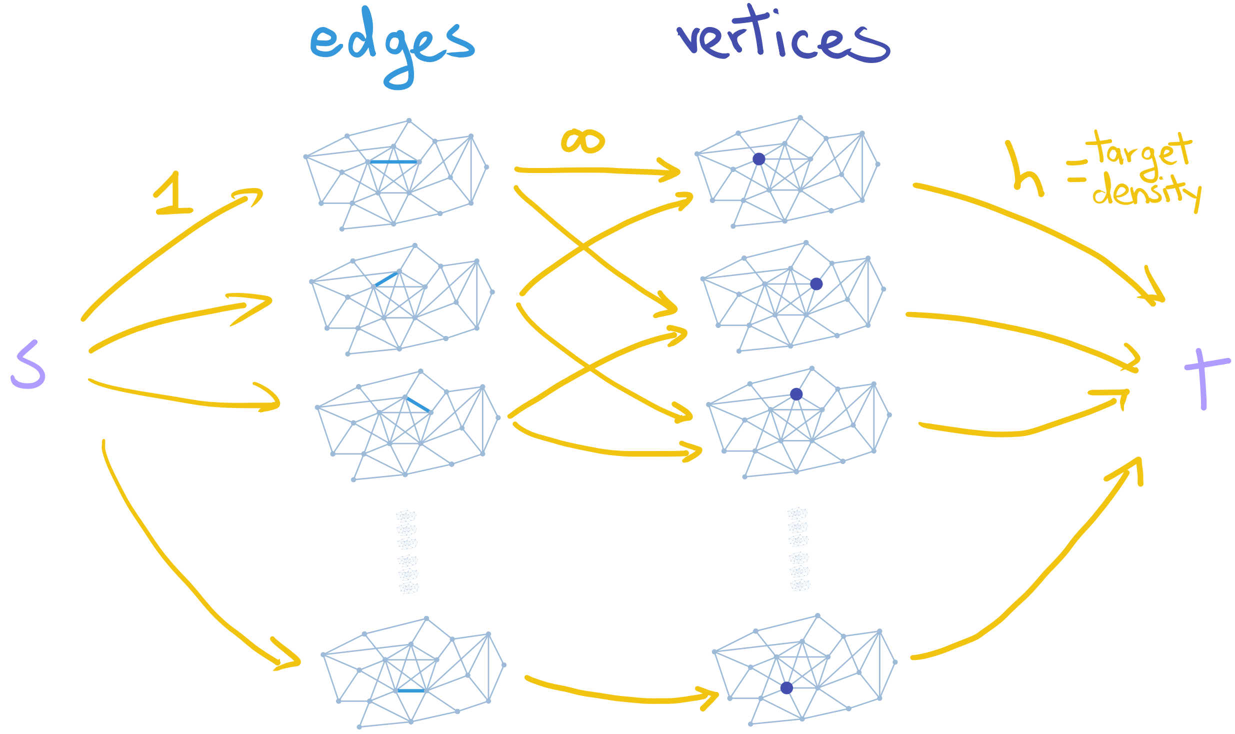
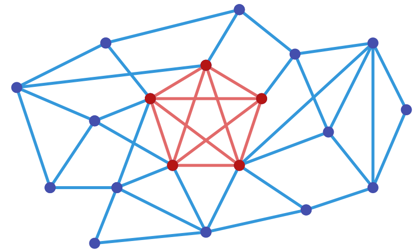


Decision version: Given $h>0$, is there a subgraph w/ density $>h$?


Densest Subgraph
edges
vertices
1
∞
h = target density
s
t

Densest Subgraph





claim: if max-flow $= m$, then all subgraphs have density $\leq h$

A subgraph of density $> \lambda$ induces (s,t)-cut of size $< m$

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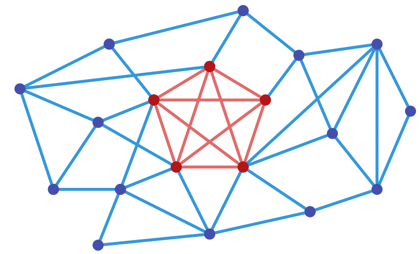

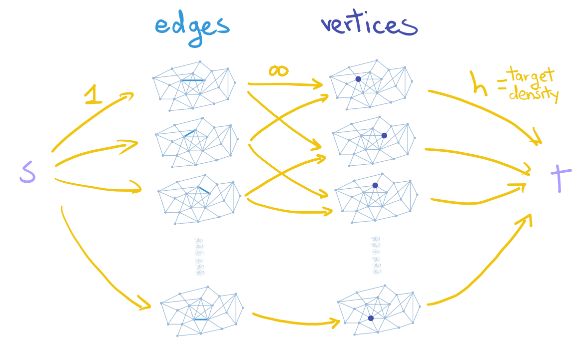

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