

Push-Relabel Max Flow

"Average" analysis of algorithms

(without sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

0, 1, 10, 11, 100, 101, ...
1111111111111111
 1000000000000000

n increments $\Rightarrow O(n)$ total time
each increment takes $O(1)$ on average

" $O(1)$ amortized time"

- 1st* bit

- 2nd bit

- 3rd bit

4

- kth bit

how often total

every time n

every 2 times $\lfloor \frac{n}{2} \rfloor$

every 4 times $\lfloor \frac{n}{4} \rfloor$

every 2^{k-1} times $\lfloor \frac{n}{2^{k-1}} \rfloor$

Amortized Analysis

"T amortized time"

$\Rightarrow$ n operations take $O(nT)$ time

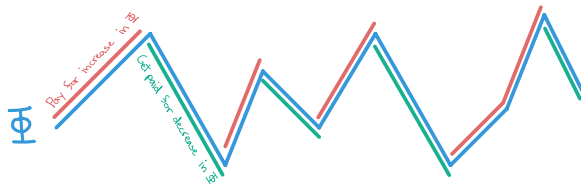
Multiple operations $OP_1, \dots, OP_K$

"OP₁ takes T_1 amortized time, OP₂ take $\approx$

OP_k takes T_k amortized time"

Potential method

Invent "potential Φ "



Amortized cost of operation will be

$$(\text{real time}) + \Delta\Phi$$

Credit schemes Idea: buy credits ahead of time to pay for work later

(running time) + $O(1) \cdot (\Delta \text{ credit}) \leq \text{amortized time}$ put: 1 credit on first bit

e.g. for binary counter, when incrementing:

when we need to flip bit i , we have already saved 1 credit to pay for it

put: 1 credit on first bit

$\frac{1}{2}$ credit on 2nd bit

$\frac{1}{4}$ credit on 3rd bit

+ $\frac{1}{2^{i-1}}$ credits on i th bit

2 credits total

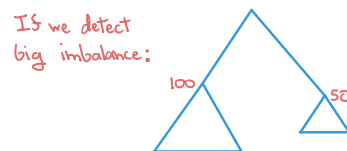
e.g. binary counter

$\Phi = \# \text{ bits set to } 1$

then for each inc,

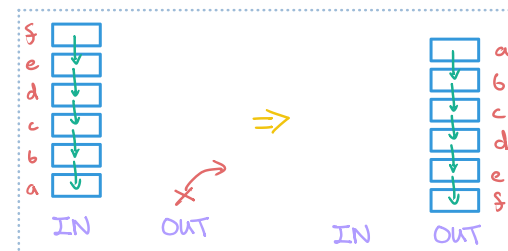
$$(\text{real time}) + \Delta \bar{\Phi} \leq O(1)$$

leach bit reset to 0 paid by $\Delta\Phi$


$$(\# \text{ in one subtree}) > 2 \times (\# \text{ in other subtree})$$

rebuild entire subtree tree optimally
(choose median to be root of subtree, recurse) (linear time)

Doubling Trick



today: push relabel algo. for max flow

- fastest algos we will see for flow

why?

(w/r/t class topics)

Roadmap for rest of semester

"Average" analysis of algorithms

↑ (w/out sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

uses amortized analysis very nicely

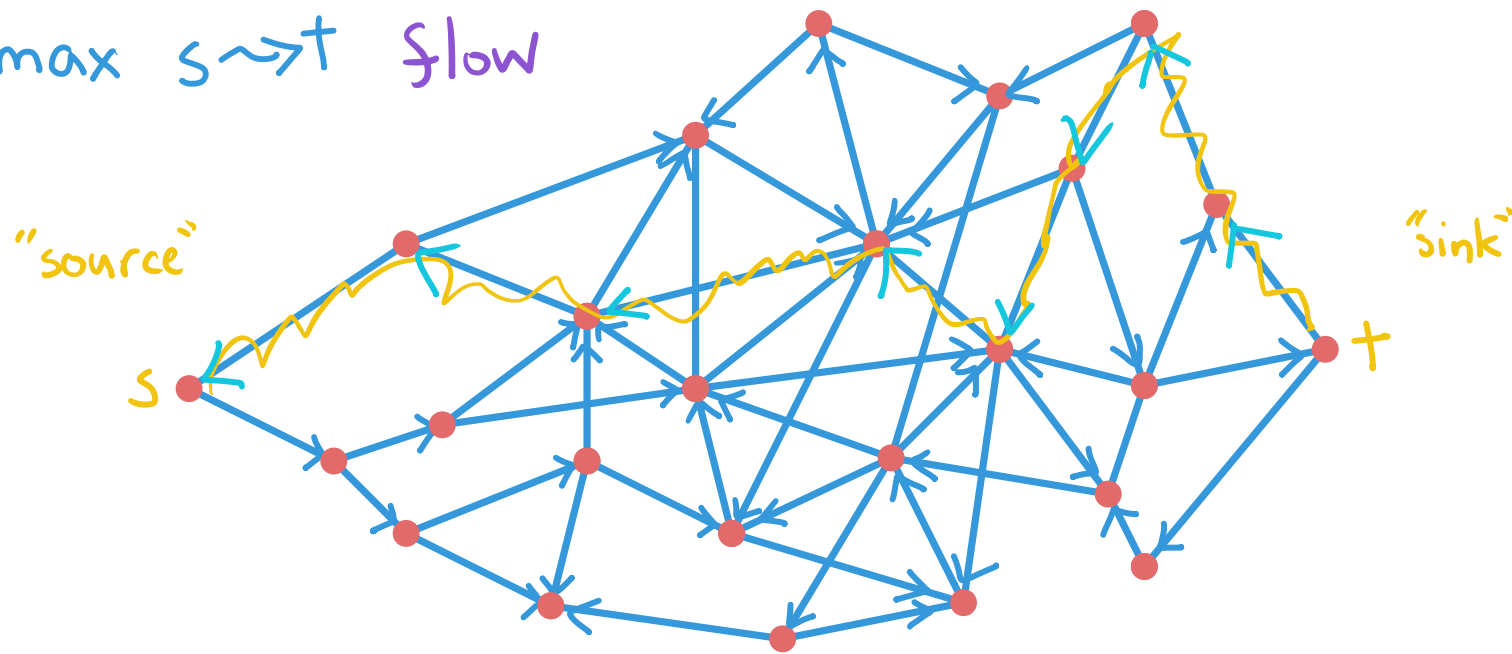
(gives us an example that is not a data structure)

(cleaned up)

(Edmonds, Karp)

How to compute

⑦ max $s \rightsquigarrow t$ flow



① while there is a path $p: s \rightsquigarrow t$

② take shortest path p and
route as much flow as possible

③ update residual graph

Analysis

Edmonds, Karp

① while there is a path $p: s \rightarrow t$

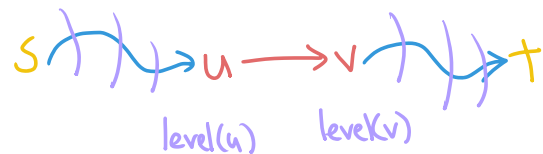
② take shortest path p and
route as much flow as possible

③ update residual graph

Key

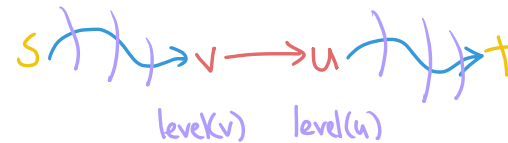
Lemma $\forall v \in V$, distance from s to v
in resid. graph never decreases

① e removed from ^(residual graph) RG,
 $\Rightarrow$ e in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(v) = \text{level}(u) + 1$$

② e = (u $\rightarrow$ v) added back in to RG
 $\Rightarrow$ (v $\rightarrow$ u) was in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(u) = \text{level}(v) + 1$$

each event ①/② $\Rightarrow$ level(u), level(v) have switched order.

level's nondecreasing by key lemma

$\Rightarrow$ e gets deleted/added at most n times

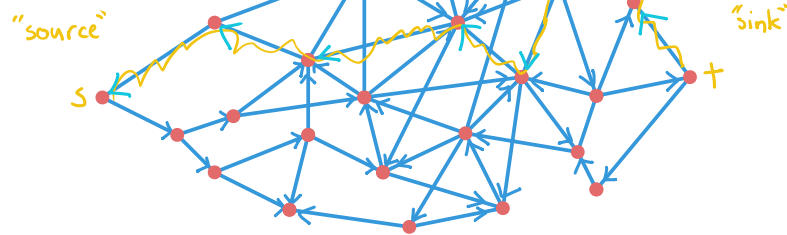
emphasis on distances in residual graph

(cleaned up)

(Edmonds, Karp)

How to compute

⑦ max $s \rightarrow t$ flow



① while there is a path $p: s \rightarrow t$

② take shortest path p and
route as much flow as possible

③ update residual graph

Analysis

Edmonds, Karp

① while there is a path $p: s \rightarrow t$

② take shortest path p and
route as much flow as possible

③ update residual graph

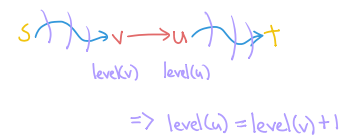
Key

Lemma $\forall v \in V$, distance from s to v
in resid. graph never decreases

① e removed from RG
 $\Rightarrow e$ in shortest $s \rightarrow t$ path



② $e = (u \rightarrow v)$ added back in to RG
 $\Rightarrow (v \rightarrow u)$ was in shortest $s \rightarrow t$ path

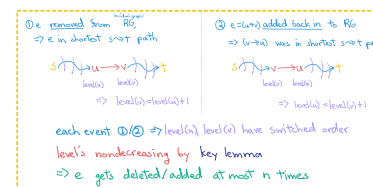
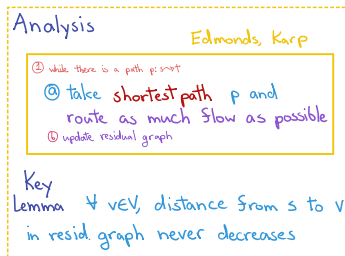
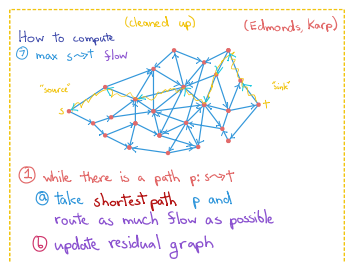


each event ①/② $\Rightarrow$ level(u), level(v) have switched order.

level's nondecreasing by key lemma

$\Rightarrow e$ gets deleted/added at most n times

emphasis on distances in residual graph



emphasis on distances in residual graph

Push-Relabel Framework

components:

① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\leftarrow$ Relaxation of flows

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\leftarrow$ Relaxation of distances in residual graph

Push-Relabel Framework

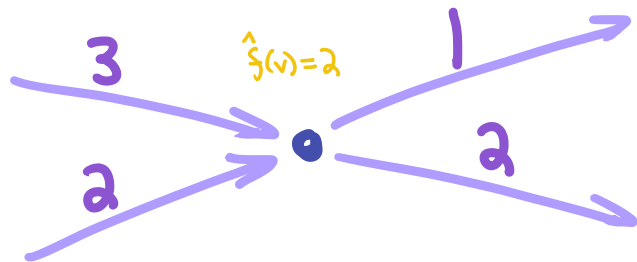
① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
↖ Relaxation of flows

- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(u,v) - \sum_{e \in \mathcal{I}^+(v)} f(v,u)$



net flow at v

② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$

↖ Relaxation of distances in residual graph

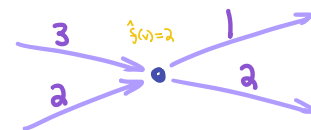
Push-Relabel Framework

Residual graph $G_f = (V, E_f)$

just like for flows:

$$\underbrace{c_f(u,v)}_{\text{residual capacity}} = c(u,v) - f(u,v) + f(v,u)$$

① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows



- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \vec{E}(v)} f(u,v) - \sum_{e \in \vec{E}^+(v)} f(v,u)$
net flow at v

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\uparrow$ Relaxation of distances in residual graph

Push-Relabel Framework

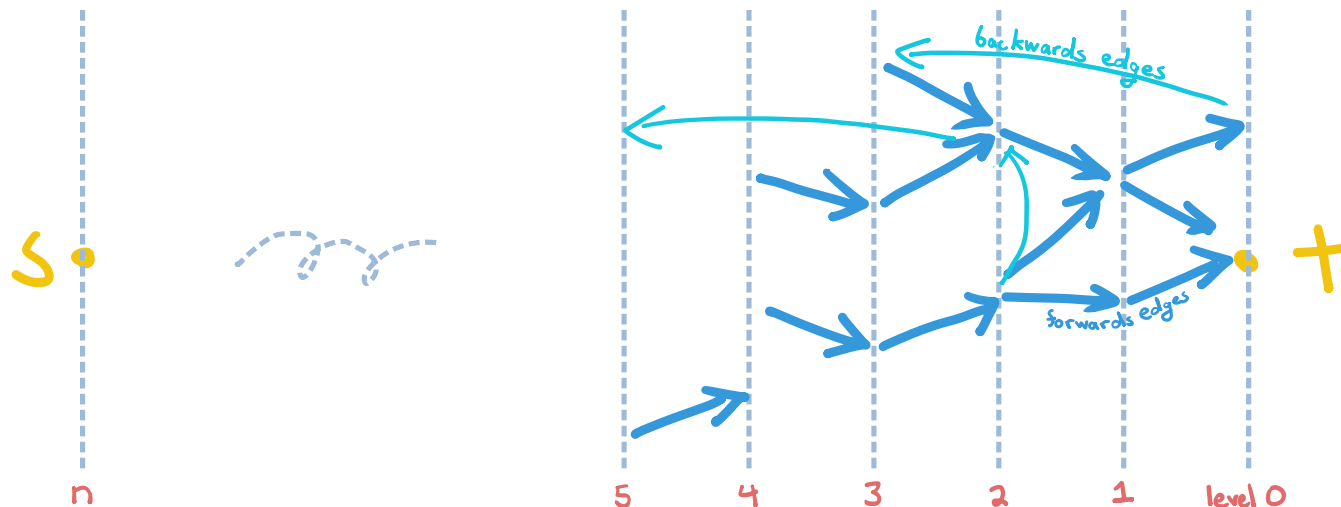
① A "preflow" $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows
 • capacity constraints $0 \leq \hat{f}(e) \leq c(e) \quad \forall e \in E$
 • nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in J(v)} \hat{f}(u,v) - \sum_{e \in I(v)} \hat{f}(v,u)$
 net flow at v

- Sink has level 0, source has level n

$$l(t) = 0, \quad l(s) = n$$

- edges (in resid. graph) go down ≤ 1 level

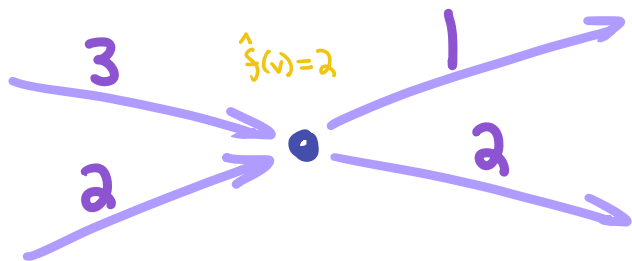
$$l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$$



Push-Relabel Framework

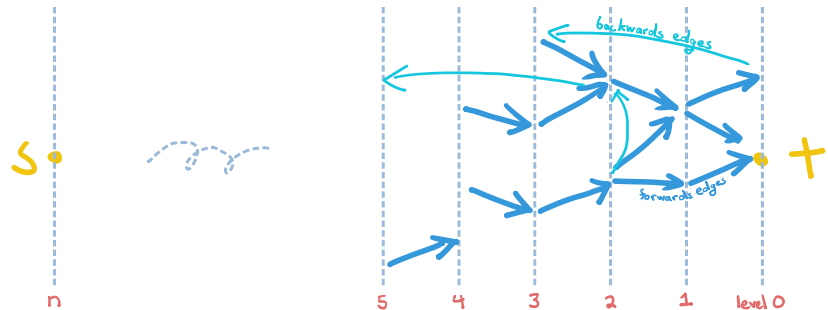
① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(u,v) - \sum_{e \in \mathcal{I}^+(v)} f(v,u)$
net flow at v



② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$



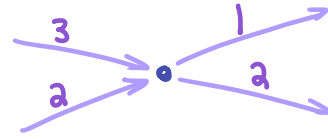
Initially: saturate all source edges, set all non-terminal levels to 0

$$f(s,v) = c(s,v) \quad \forall (s,v) \in \mathcal{I}^+(s)$$

$$\text{level}(s) = n, \quad \text{level}(v) = 0 \quad \forall v \neq t$$

call a vertex u "active" if

$$\hat{f}(u) > 0$$



high-level goal:

push flow out of u "towards t "

Push-Relabel Operations

① Push flow along forward edge

② Relabel vertex to higher level

(while following the rules)

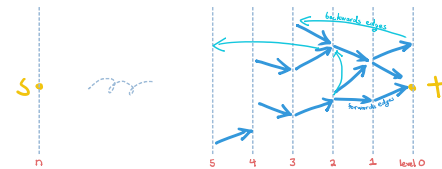
Push-Relabel Framework

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \underbrace{\sum_{e \in \mathcal{I}^+(v)} f(u,v) - \sum_{e \in \mathcal{I}^-(v)} f(v,u)}_{\text{net flow at } v}$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$



Initially:

$$f(s,v) = c(s,v) \quad \forall (s,v) \in \mathcal{I}^+(s)$$

$$\text{level}(s) = n, \quad \text{level}(v) = 0 \quad \forall v \neq t$$

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$

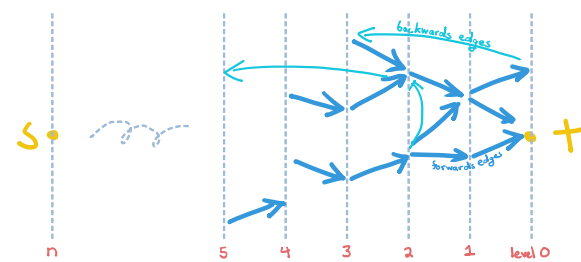
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^+(v)} f(u,v) - \sum_{e \in \mathcal{I}^-(v)} f(v,u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$



① Push flow along forward edge

$e = (u,v) \in E_f$ w/ $l(u) = l(v) + 1$

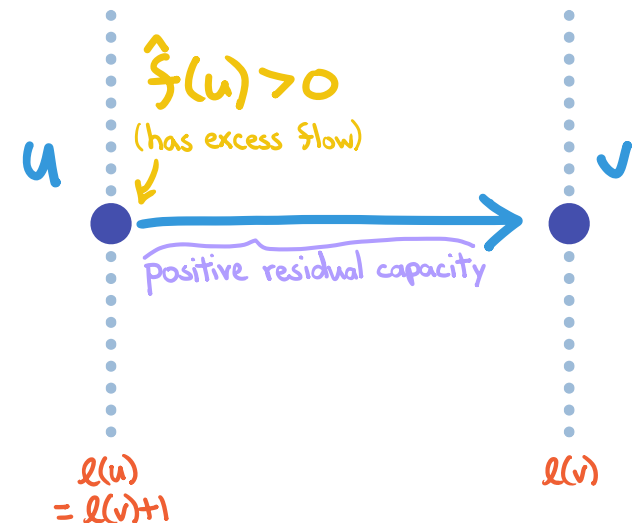
u active

(usually)

push as much flow as possible:

$$\min \{ c_f(e), \hat{f}(u) \}$$

(residual capacity) (excess flow)



② Relabel vertex to higher level

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$

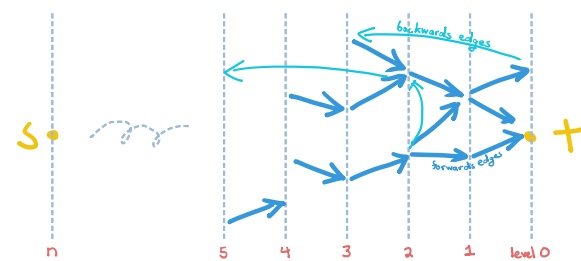
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^+(v)} f(u,v) - \sum_{e \in \mathcal{I}^-(v)} f(v,u)$

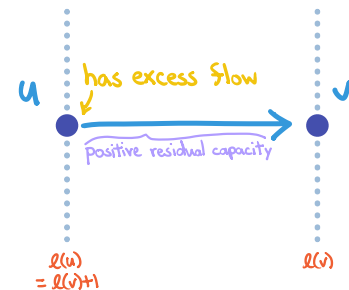
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$



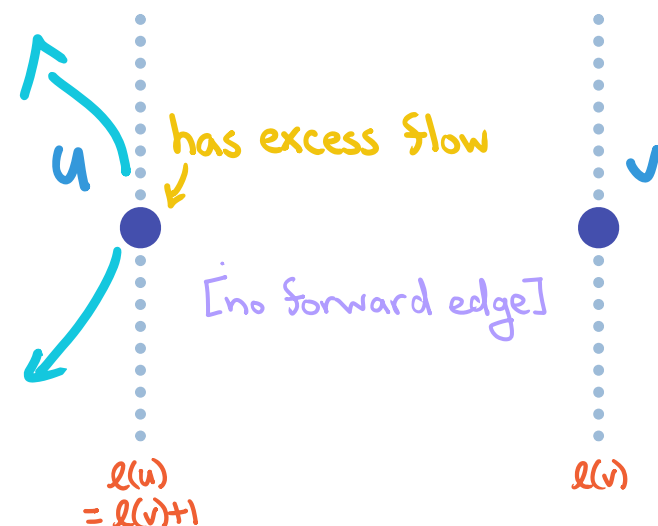
① Push flow along forward edge



② Relabel vertex to higher level

u active, no forward edge

safe to set $l(u) = l(u) + 1$



Algorithm:

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

WTF:

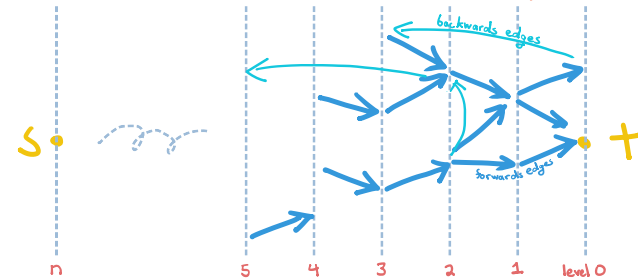
- flow? (let alone max flow?)
- running time?
- does it even terminate?

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

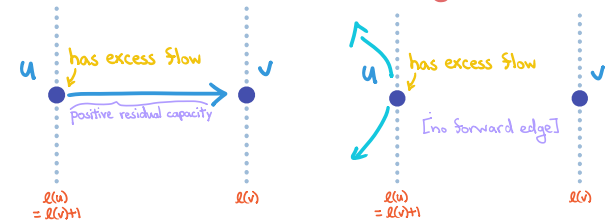
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

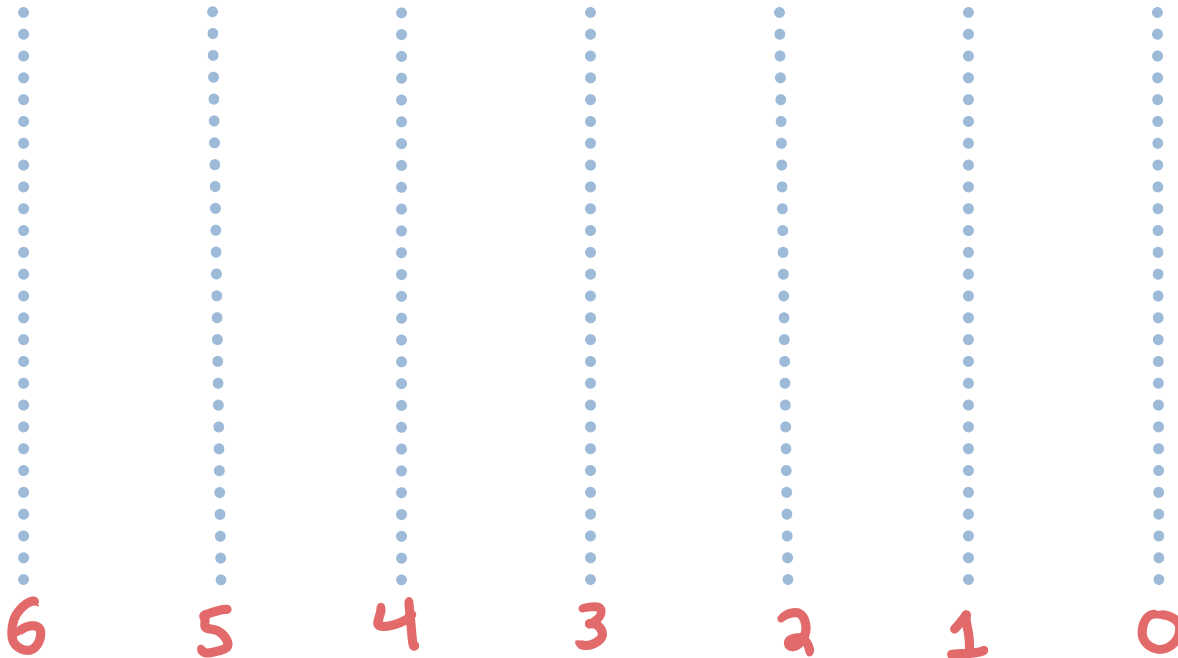
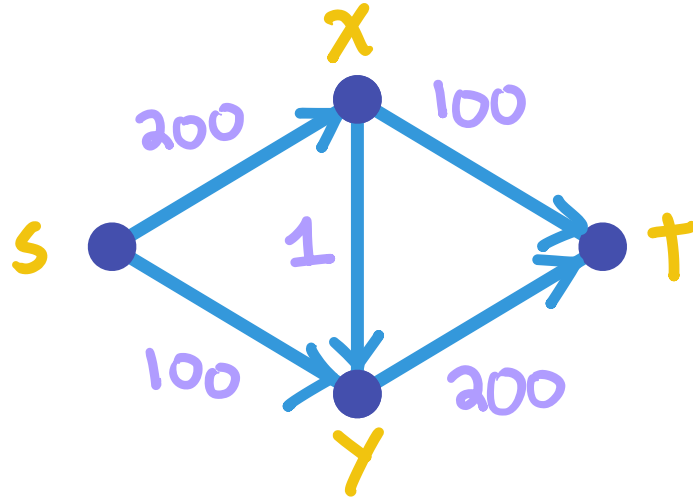
② Relabel vertex to higher level



① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v



① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

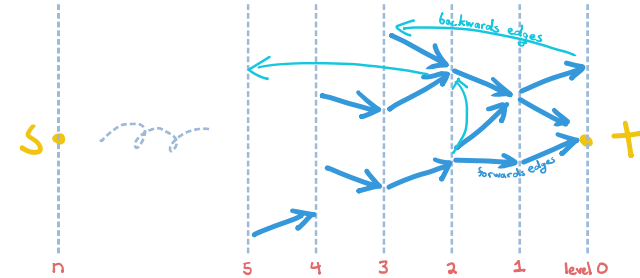
• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

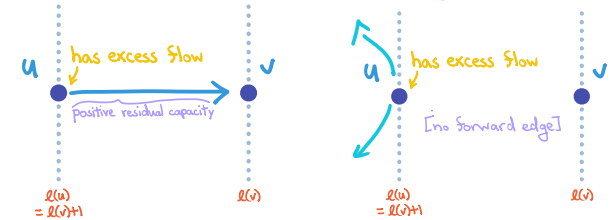
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

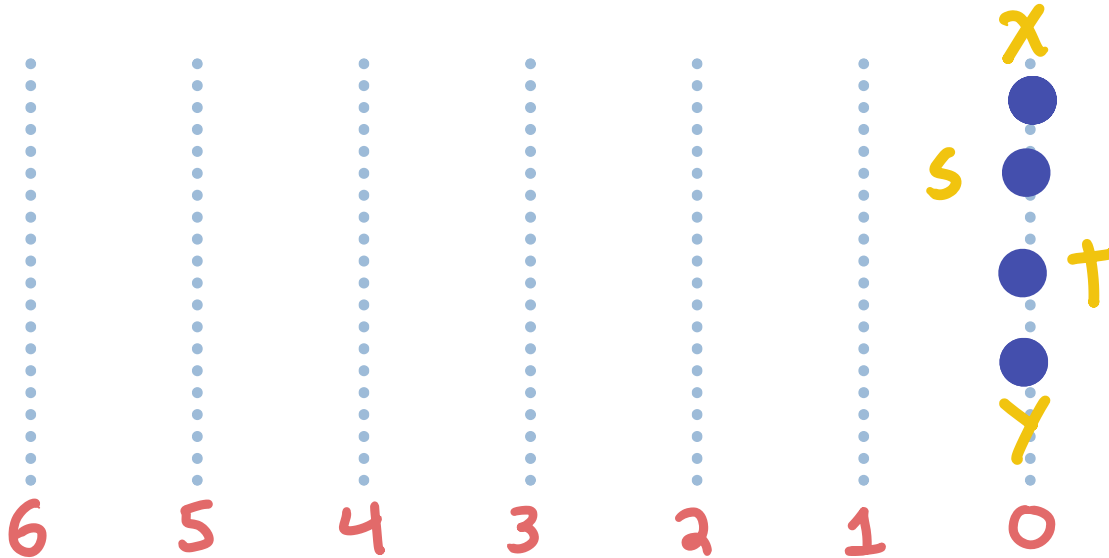
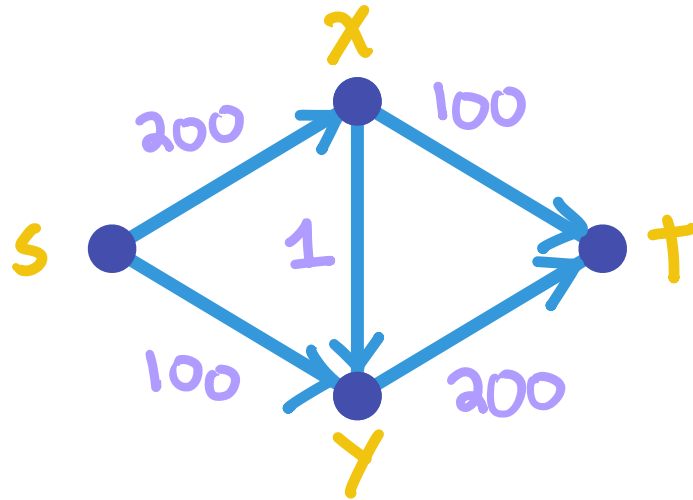
② Relabel vertex to higher level



① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v



① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

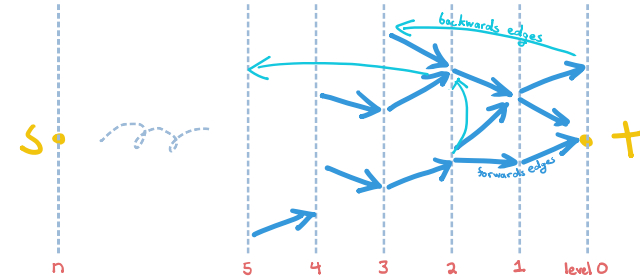
$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in I(v)} f(u, v) - \sum_{e \in O(v)} f(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

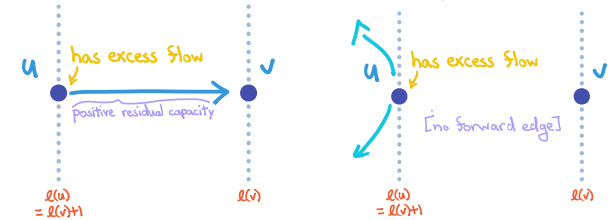
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Algorithm:

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

WTF:

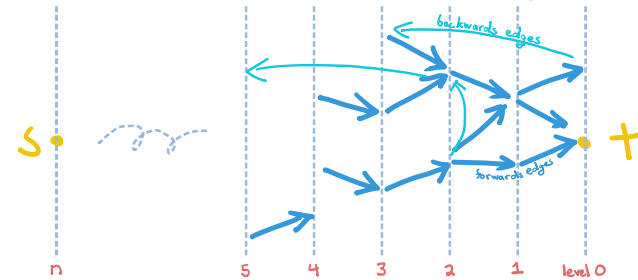
- flow? (let alone max flow?)
- running time?
- does it even terminate?

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

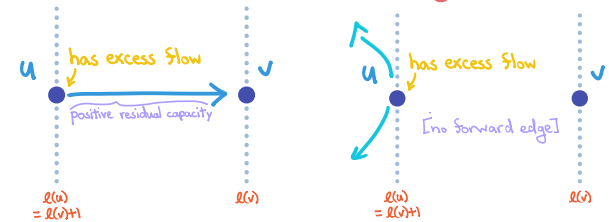
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Understanding "preflows"

Lemma let f be a preflow.
then f is a path packing w/
 $\hat{f}(v)$ (s,v) -paths for all v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(u,v) - \sum_{e \in \mathcal{I}^+(v)} f(v,u)$

Understanding "labels"

Lemma $\forall x, y \in V,$

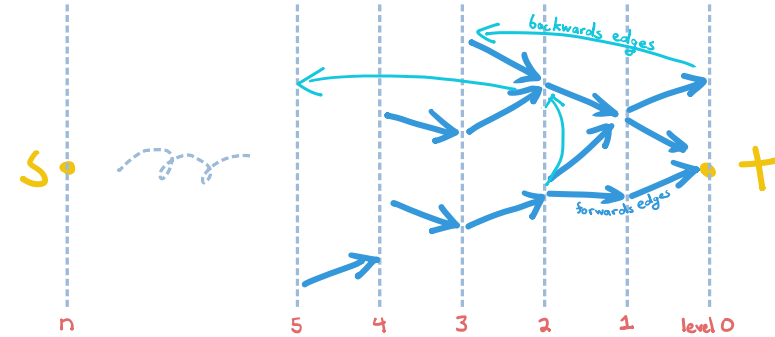
$$l(x) - l(y) \leq d(x, y)$$

where $d(x, y)$ = distance in resid. graph

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



Understanding "labels"

Lemma $\forall x, y \in V,$
 $l(x) - l(y) \leq d(x, y)$

where $d(x, y)$ = distance in resid. graph

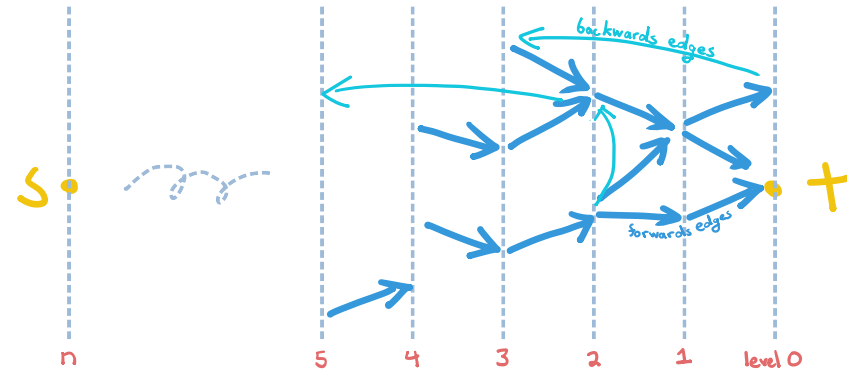
$$\Rightarrow d(s, t) \geq l(s) - l(t) = n$$

Corollary: No (s, t) -path in residual graph!

(always an (s, t) -cut in residual graph
separated by empty layers)

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



Understanding "preflows"

Lemma let f be a preflow.
then f is a path packing w/
 $\hat{f}(v)$ (s,v) -paths for all v

Understanding "labels"

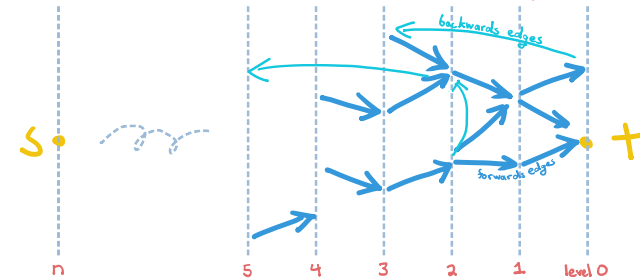
Lemma $\forall x, y \in V$,
 $l(x) - l(y) \leq d(x, y)$
where $d(x, y)$ = distance in resid. graph

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

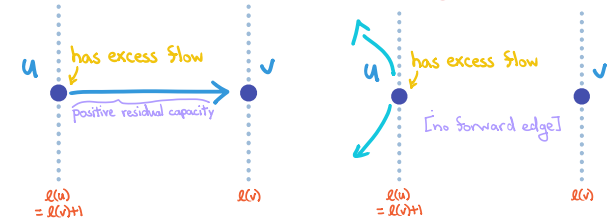
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Lemma if v active, then
 $l(v) < 2n$

Lemma f preflows, $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels

if f is a flow, then

① f is a max flow.

② there is empty layer $i \in \{1, \dots, n-1\}$

s.t. for $U = \{v: l(v) > i\}$,

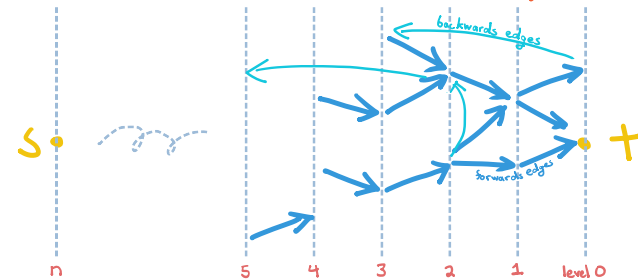
$\partial^+(U)$ is a min (s, t) -cut

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \partial^-(v)} f(e) - \sum_{e \in \partial^+(v)} f(e)$

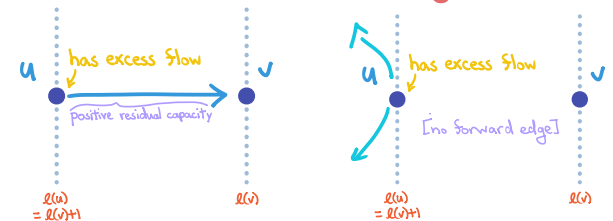
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Lemma f preflows $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels

if f is a flow, then

① f is a max flow.

② there is empty layer $i \in \{1, \dots, n-1\}$

s.t. for $U = \{v: l(v) > i\}$,

$\partial^+(U)$ is a min (s,t) -cut

(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

when no v 's are active,
 f is a max flow.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

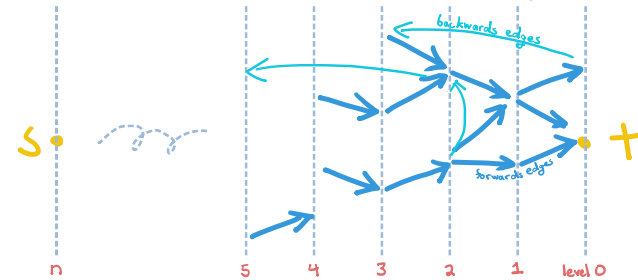
• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(e) - \sum_{e \in \mathcal{I}^+(v)} f(e)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

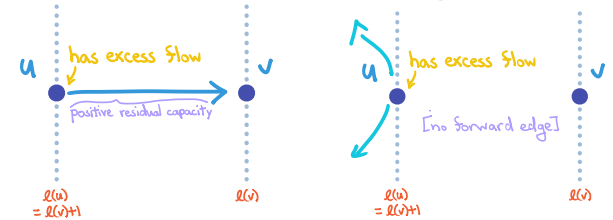
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Now we know why algo is correct.
next: running time

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

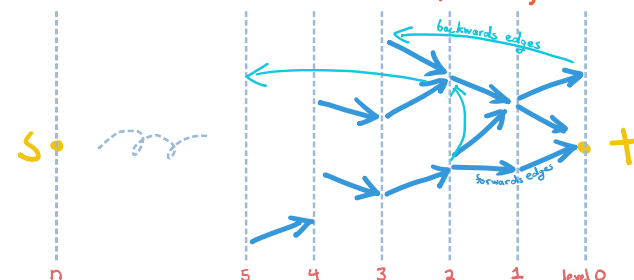
• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

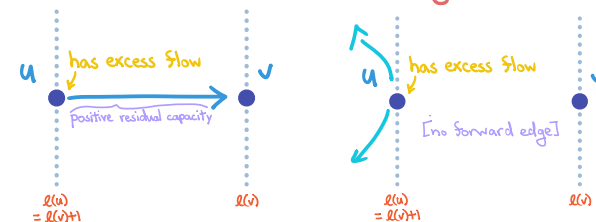
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Two kinds of pushes:

① Saturating push

uses all capacity of edge

② Non-saturating push

instead pushes out all excess flow from v

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

• capacity constraints

$$0 \leq \hat{f}(e) \leq c(e) \quad \forall e \in E$$

• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

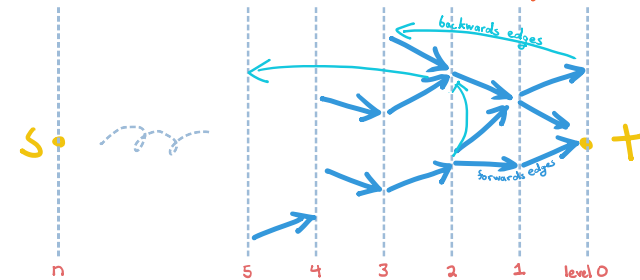
$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} \hat{f}(u, v) - \sum_{e \in \mathcal{O}(v)} \hat{f}(v, u)$$

② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$

$\sim$ Relaxation of distances in residual graph

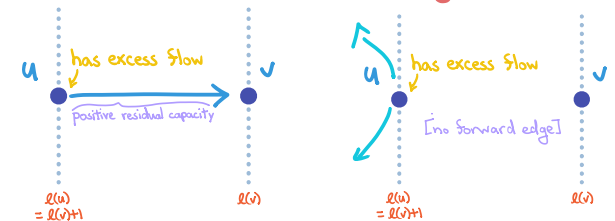
• sink has level 0, source has level n
 $\ell(t) = 0, \ell(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $\ell(v) \geq \ell(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

- ① while there is active $v \in V - \{s, t\}$
 - Ⓐ try to push flow out of v
 - Ⓑ otherwise relabel v

we will bound:

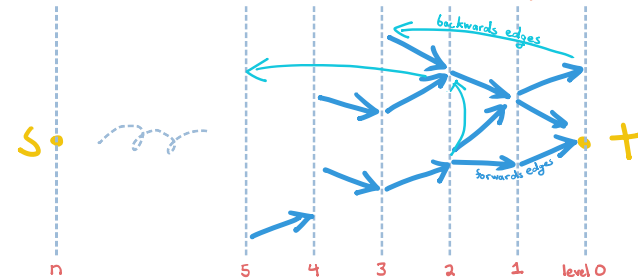
- ① # relabelings
- ② # saturating push
- ③ # non-saturating pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

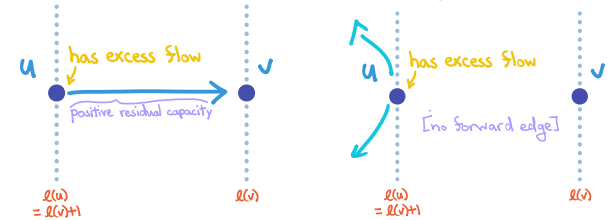
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

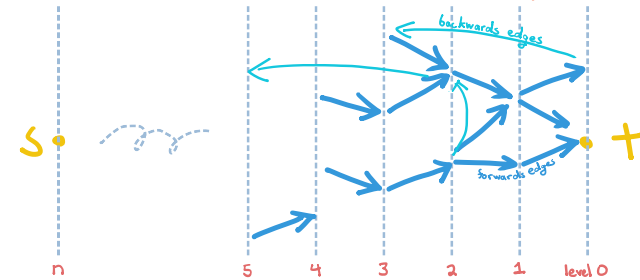
- ① while there is active $v \in V - \{s, t\}$
- ② try to push flow out of v
- ③ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

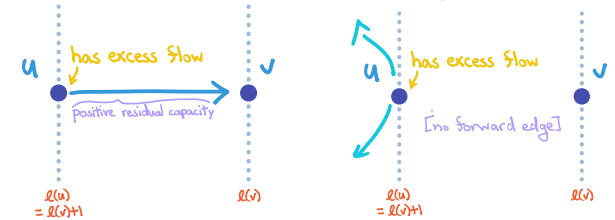
- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma

(# saturating pushes) $\leq O(mn)$

($O(n)$ per edge)

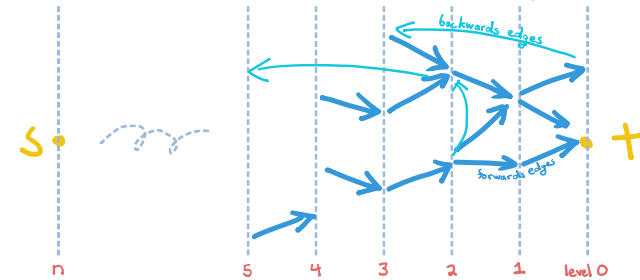
- ① while there is active $v \in V - \{s, t\}$
- ② try to push flow out of v
- ③ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

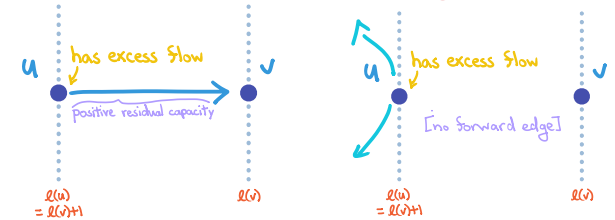
- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

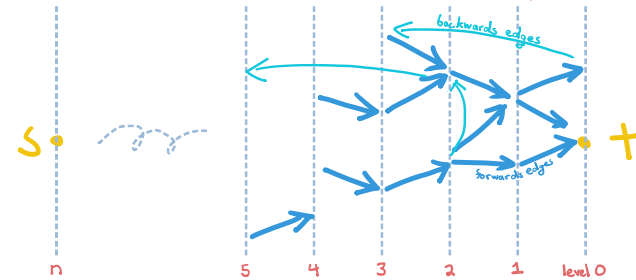
- ① while there is active $v \in V - \{s, t\}$
- ② try to push flow out of v
- ③ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

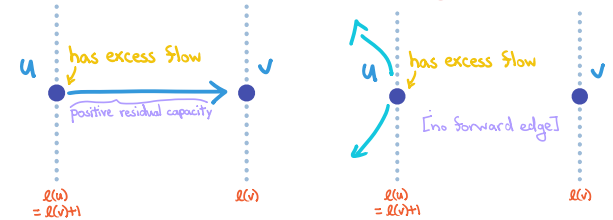
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

ⓐ try to push flow out of v

ⓑ otherwise relabel v

$$(\# \text{ relabels}) \leq O(n^2)$$

$$(\# \text{ saturating pushes}) \leq O(mn)$$

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

+

$$O(mn^2)$$

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

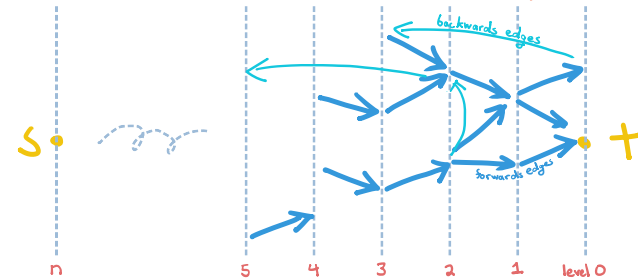
$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^+(v)} f(u, v) - \sum_{e \in \mathcal{I}^-(v)} f(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

$\sim$ Relaxation of distances in residual graph

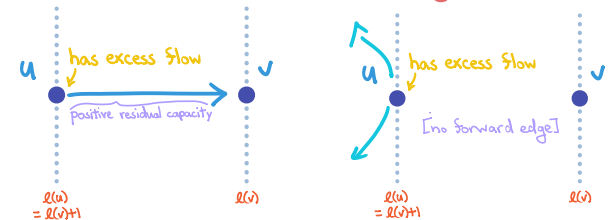
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

 ⓐ try to push flow out of v

 ⓑ otherwise relabel v

$O(mn^2)$ time

faster?

Bottleneck was non-saturating pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

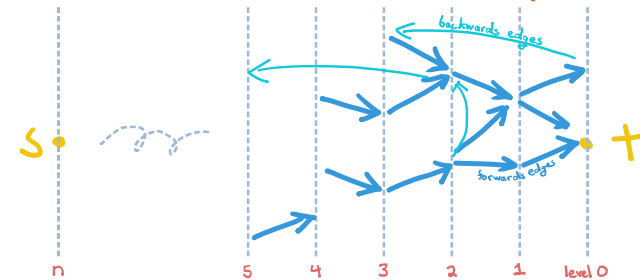
$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

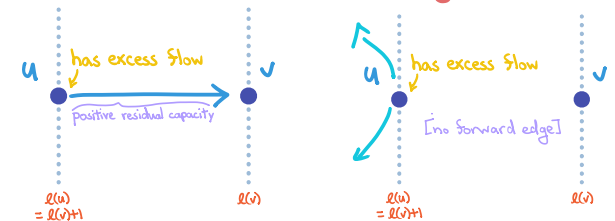
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

① while there is active $v \in V - \{s, t\}$

★ let $v =$ active w/ highest level

@ try to push flow out of v

⑥ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

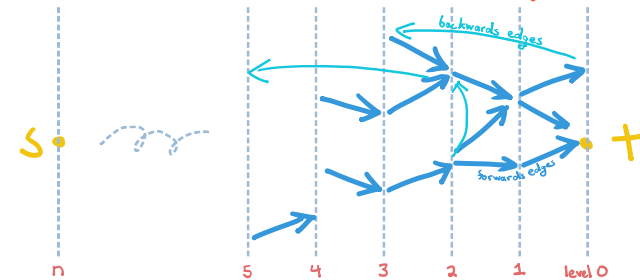
$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

$\sim$ Relaxation of distances in residual graph

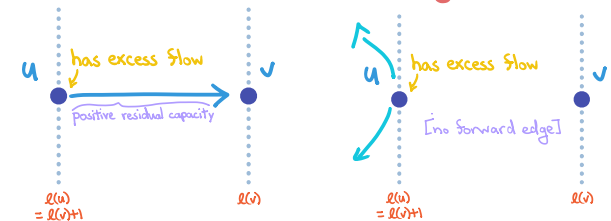
• sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$

• edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v = \text{active w/ highest level}$
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

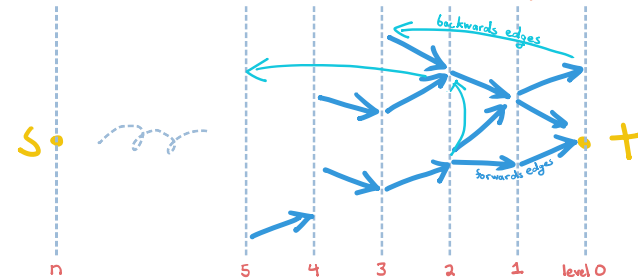
↑ better?

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

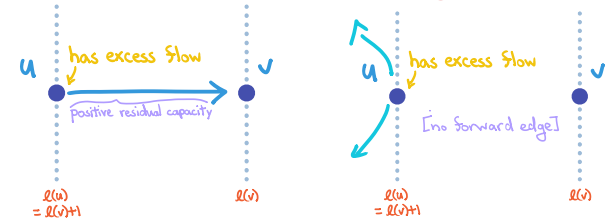
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

proof.

suppose non satur. push from v

Top-down push-relabel

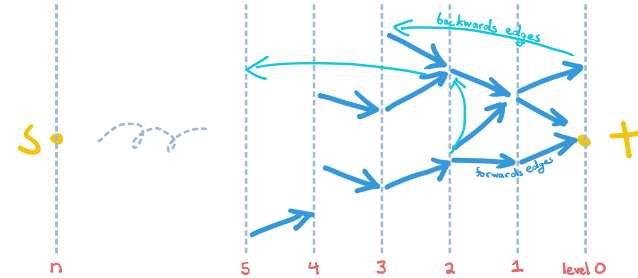
- ① while there is active $v \in V - \{s, t\}$
 - ⊛ let v = active w/ highest level
 - ⊙ try to push flow out of v
 - Ⓛ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

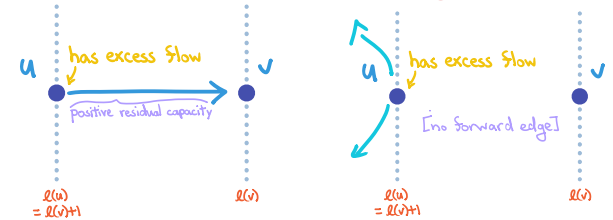
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v =$ active w/ highest level
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

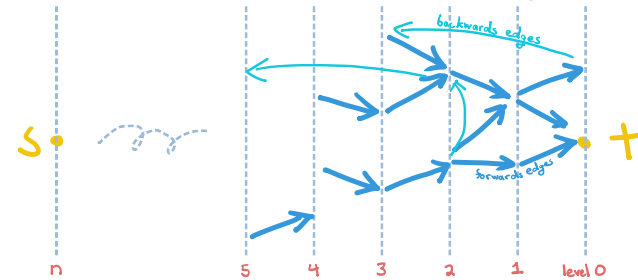
$\Rightarrow O(n^3)$ time overall.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

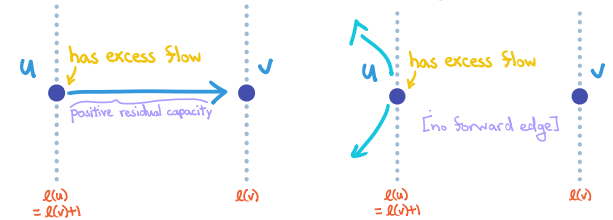
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



(better?)

Lemma (# non-satur. pushes) $\leq O(n^2 \sqrt{m})$

proof. $\Phi = \sum_{v \text{ active}} |w: l(w) \leq l(v)|$

relabel, satur. push: $\Delta \Phi \text{ by } \leq n$
 $\Rightarrow$ total increase of $O(mn^2)$

non-satur. push at v : $\Delta \Phi = -|w: l(w) = l(v)|$

"big" if $|w: l(w) = l(v)| \geq k$ w/ $k \text{ TBD}$

"small" if $|w: l(w) = l(v)| \leq k$

claims: $O(mn^2/k)$ big pushes

$O(n^2 k)$ small pushes

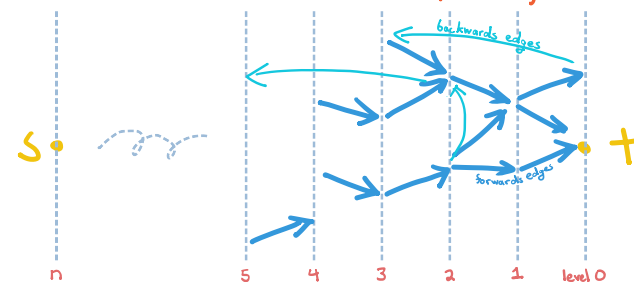
Then set $k = \sqrt{m}$

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

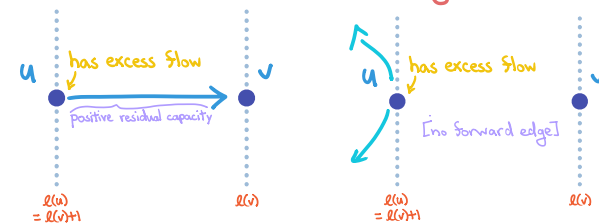
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u,v) - \sum_{e \in \mathcal{O}(v)} f(v,u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ⊛ let v = active w/ highest level
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma (# non-satur. pushes) $\leq O(n^2 \sqrt{m})$

proof. $\Phi = \sum_{v \text{ active}} |w: l(w) \leq l(v)|$

relabel, satur. push: $\Delta \Phi \leq n$
 $\Rightarrow$ total increase of $O(mn^2)$

non-satur. push at v : $\Delta \Phi = -|w: l(w) = l(v)|$

"big" if $|w: l(w) = l(v)| \geq k$ w/ k TBD

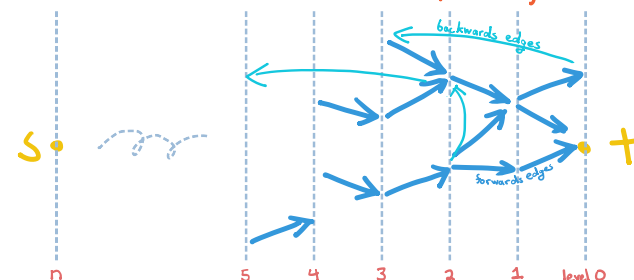
claim: $O(mn^2/k)$ big pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

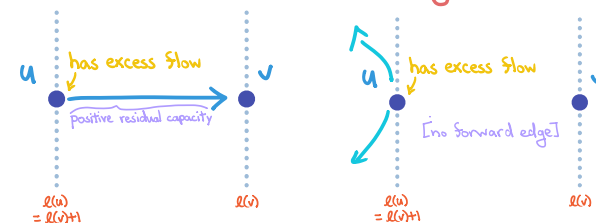
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u,v) - \sum_{e \in \mathcal{O}(v)} f(v,u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u,v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ⊛ let v = active w/ highest level
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma (# non-satur. pushes) $\leq O(n^2 \sqrt{m})$

proof. $\Phi = \sum_{v \text{ active}} |w: \ell(w) \leq \ell(v)|$

relabel, satur. push: $\Delta \Phi \leq n$
 $\Rightarrow$ total increase of $O(mn^2)$

non-satur. push at v : $\Delta \Phi = -|w: \ell(w) = \ell(v)|$

"small" if $|w: \ell(w) = \ell(v)| \leq k$ w/ k TBD

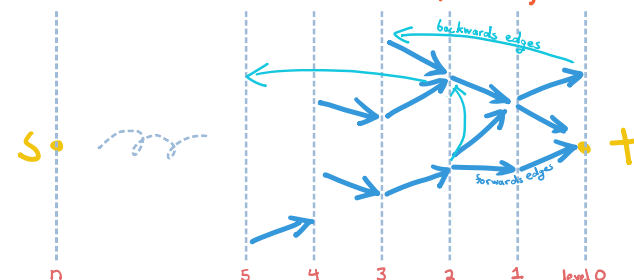
claim: $O(n^2 k)$ small pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

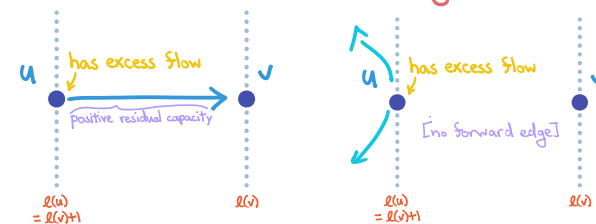
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u,v) - \sum_{e \in \mathcal{O}(v)} f(v,u)$

② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $\ell(t) = 0, \ell(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $\ell(v) \geq \ell(u) - 1 \quad \forall (u,v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ⊛ let v = active w/ highest level
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v